MIXED BROWNIAN–FRACTIONAL BROWNIAN MODEL: ABSENCE OF ARBITRAGE AND RELATED TOPICS

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Let $W = (W_t)_{t\geq 0}$ be a standard Wiener process and $B^H = (B_t^H)_{t\geq 0}$ be a fractional Brownian motion with the Hurst index H, $H \in \left(\frac{1}{2}, 1\right)$, both defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$.

Consider a mixed version of the Black-Merton-Scholes model, i.e., a $\left(B, S\right)$-market with a bond B and a stock S, where

$$B_t = e^{rt}, \quad S_t = e^{aW_t + bB_t^H + ct}, \quad r, a, b, c \in R. \tag{1}$$

For a given strategy (or a portfolio) $\pi = (\beta_t, \gamma_t)_{t\geq 0}$ the capital $X = (X_t)_{t\geq 0}$ corresponding to this portfolio equals

$$X_t = B_t \cdot \beta_t + S_t \cdot \gamma_t. \tag{2}$$

We make the following assumptions about the strategy π:

1) π is self-financing, i.e.,

$$X_t = X_0 + \int_0^t \beta_s \, dB_s + \int_0^t \gamma_s \, dS_s; \tag{3}$$

2) π is of Markov type, i.e.,

$$\beta_t = \beta\left(S_t, t\right), \quad \gamma_t = \gamma\left(S_t, t\right). \tag{4}$$

We say that the strategy π has an arbitrage opportunity if there exists $T > 0$ such that

$$X_0 = 0, \quad X_T \geq 0 \ (P - a.s.), \quad P(X_T > 0) > 0.$$

It is well-known that in the case of standard Black-Merton-Scholes model, i.e., when $b = 0$ in (1), there exists an equivalent martingale measure. This fact guarantees the absence of arbitrage. In the case of pure fractional model, i.e., when $a = 0$ in (1), the arbitrage possibility depends on the definition of stochastic integral. It is easy to give arbitrage examples in continuous time trading with self-financing strategies, if one uses the Riemann-Stieltjes integral. It was demonstrated by Rogers [1], Shiryaev [2], Dasgupta [3]. If one uses the Wick-Itô-Skorohod integral she obtains an arbitrage-free model. Pioneer papers on this topics belong to Hu, Øksendal [4] and Elliott, van der Hoek [5], then the situation was carefully studied by Bender [6] where some clarifications were introduced. The results have been summarised by Sottinen and Valkeila in [7]. In the mixed model (1) with $a \neq 0$ and $b \neq 0$ some results were obtained in the papers of Kuznetsov [8], Cheridito [9] and Mishura, Valkeila [10]. Kuznetsov proved the absence of arbitrage under condition of independence of processes W and B^H. Cheridito established that for $H \in \left(\frac{3}{4}, 1\right)$ the

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mixed model with independent W and B^H is equivallent to the one with Brownian motion and hence it is arbitrage-free. Conversely, the work of Mishura and Valkeila guarantees the absence of arbitrage when the trajectories of process B^H are in some sense the functionals of trajectories of process W.

The main result of our paper is that the mixed market is arbitrage-free for any W and B^H if we restrict ourselves with self-financing Markov-type smooth strategies.

We introduce for $\alpha > 0$ the process $B^{H,\alpha}$ which is continuously differentiable and for which convergence $\mathrm{E}\left|B_t^H - B_t^{H,\alpha}\right|^m \to 0$ holds as $\alpha \to 0+$ for $2 \le m < \frac{1}{1-H}$. Also we prove in the next section that if process f belongs to the class $C^{2(1-H)+\varepsilon}$ with some $\varepsilon > 0$, then for $H \in \left(\frac{3}{4}, 1\right)$ convergence in probability $\int_0^T f(u)\, dB_u^{H,\alpha} \to \int_0^T f(u)\, dB_u^H$ holds as $\alpha \to 0+$, where the last integral is the Riemann-Stieltjes one. Using this auxiliary result we represent the process of capital in a mixed Black–Merton–Scholes model (1) as a limit of some semimartingales.

References

[1] Rogers L.C.G. (1997) Arbitrage with fractional Brownian motion, *Mathematical Finance* **7**, 95–105.

[2] Shiryaev A.N. (1998) On Arbitrage and Replication for Fractal Models, *Proc. Workshop on Mathematical Finance*, *Paris: INRIA*, 1–7.

[3] Dasgupta A. (1998) Fractional Brownian Motion: Its Properties and Applications to Stochastic Integration, *Ph. Thes., Dept. of Statistics, Univ. of North Carolina.*

[4] Hu Ya., Øksendal B. (2003) Fractional white noise calculus and applications to finance, *Infinite Dimentional Analysis, Quantum Probability and Related Topics*, **6** no. 1, 1–32.

[5] Elliot R.J., van der Hoek J. (2003) A general Fractional Noise Theory and Applications to Finance, *Mathematics Finance*, **13** no. 2, 301–330.

[6] Bender C. (2003) Integration with respect to a Fractional Brownian Motion and Related Market Models, *PhD thesis, Konstanz, Germany, Hartung-Gorre Verlag Konstanz.*

[7] Sottinen T., Valkeila E. (2003) On arbitrage and replication in the fractional Black-Scholes pricing model, *Statistics and Decisions* **21**, 93–107.

[8] Kuznetsov Yu.A. (1999) The Absence of Arbitrage in a Model with Fractional Brownian Motion, *Usp. Mat. Nauk* **54** no. 4, 175–176 (In Russian).

[9] Cheridito P. (2001) Regularizing Fractional Brownian Motion with a View towards Stock Price Modelling, *PhD thesis, Zurich.*

[10] Mishura Yu.S., Valkeila E. (2002) The Absence of Arbitrage in a Mixed Brownian-Fractional Brownian Model, *Proc. of the Steklov Institute of Mathematics* **237**, 215–224.

[11] Norros I., Valkeila E. and Virtamo J. (1999) An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions, *Bernoulli* **55**, 571–587.

[12] Protter P. (1990) Stochastic integration and differential equations, *Springer-Verlag*, 302 p.

[13] Zähle M. (1998) Integration with respect to fractal functions and stochastic calculus. Part I, *Probab. Theory Relat. Field* **111**, 333–372.

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