Tail bounds for the joint distribution of the surplus prior to and at ruin

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Abstract

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We consider the classical risk model where claims $Y_1, Y_2, \ldots$ arrive in a compound Poisson process with rate $\lambda$. The claims are independent identically distributed non-negative random variables and have common distribution function $P$ with finite mean $\mu$. In the case where $P$ has a density, we denote this density by $p$. We further assume that the claims are independent of the claim-arrivals process. Premiums are paid to the insurer continuously at a rate $c$ per unit time. The surplus of the insurer at time $t$ is then $U(t) = u + ct - \sum_{k=1}^{N_t} Y_i$, where $u$ is the initial surplus and $N_t$ is the number of claims until $t$. We assume throughout that $c > \lambda \mu$, so that ruin is not certain to occur. Moreover, we write $c = (1 + \theta) \lambda \mu$, where $\theta$ is the relative security loading. Let $T$ denote the time of ruin, i.e. the time that the surplus becomes negative for the first time and note that $T$ is a defective random variable. The probability of ruin is then defined by

$$\psi(u) = P(T < \infty | U(0) = u).$$ (1)

The tail for the joint distribution of the surplus prior to and at ruin is defined by

$$H(u, x, y) = P(U(T) > x, |U(T)| > y, T < \infty | U(0) = u).$$ (2)

For $x = 0$ in (2), we define

$$\overline{H}(u, y) = P(|U(T)| > y, T < \infty | U(0) = u),$$ (3)

which is the tail of the defective distribution of the deficit at ruin, while for $x = y = 0$ in (2) we recover the probability of ruin $\psi(u)$. The main purpose of the present paper is to obtain new, lower and upper, bounds for the function $H(u, x, y)$ in (2). Let $P_e$ be the equilibrium distribution associated with the claim size distribution $P$, defined for $x \geq 0$ by $P_e(x) = \mu^{-1} \int_0^x \bar{P}(t) dt$. Here, and in the following, $\overline{P}(x) = 1 - P(x)$ denotes the tail of a distribution. Define also $H(u) = 1 - \psi(u)$ to be the probability of non-ruin starting with a capital $u$. First we note the following, which gives an exact representation for $\overline{H}(u, x, y)$ in the case $u > x$.

**Proposition 1.** For $u > x$ and $y \geq 0$, it holds that

$$\overline{H}(u, x, y) = \frac{\phi}{1 - \phi} \left( \int_0^{u-x} \overline{P}_e(u + y - z) dH(z) + \overline{P}_e(x + y) [\psi(u - x) - \psi(u)] \right) + \phi \overline{P}_e(u + y).$$ (4)
In the sequel, we obtain better lower and upper bounds for the function \( \overline{H}(u, x, y) \) is given by

\[
\overline{H}(u, x, y) \leq \frac{\phi}{1 - \phi} \left( \frac{1}{1} \overline{F}_e(x + y)[\phi - \psi(u)] + \phi \overline{F}_e(u + y) \right).
\]

while a lower bound is

\[
\overline{H}(u, x, y) \geq \frac{\phi}{1 - \phi} \left( \overline{F}_e(u + y)[1 - \psi(u - x)] + \overline{F}_e(x + y)[\psi(u - x) - \psi(u)] \right).
\]

In the sequel, we obtain better lower and upper bounds for the function \( \overline{H}(u, x, y) \) using numerical methods based on a partition interval \((0, u - x]\). Denote \( d_n = (u - x)/n \).

**Theorem 1.** For \( u > x \) and \( y \geq 0 \), it holds that

\[
\overline{H}(u, x, y) \leq \frac{\phi}{1 - \phi} \left\{ \sum_{k=1}^{n} \left( \overline{F}_e(u + y - k d_n) - \overline{F}_e(u + y - (k - 1) d_n) \right) \int_{(k-1)d_n}^{kd_n} \psi(z) \, dz 
- \overline{F}_e(x + y) \psi(u - x) + \overline{F}_e(u + y) \right\},
\]

where

\[
z_{k,n} = \frac{(u - x)[k p_e(u + y - k d_n) - (k - 1)p_e(u + y - (k - 1) d_n)]/n}{p_e(u + y - k d_n) - p_e(u + y - (k - 1) d_n)}
- \frac{\overline{F}_e(u + y - k d_n) - \overline{F}_e(u + y - (k - 1) d_n)}{p_e(u + y - k d_n) - p_e(u + y - (k - 1) d_n)}
\]

for \( k = 1, 2, \ldots, n, \) and \( z_{0,n} = 0, z_{n+1,n} = u - x \).

**Theorem 2.** For \( u > x \) and \( y \geq 0 \), it holds that

\[
\overline{H}(u, x, y) \geq \frac{\phi}{1 - \phi} \left\{ \sum_{k=1}^{n+1} p_e(u + y - (k - 1) d_n) \int_{z_{k-1,n}}^{z_{k,n}} \psi(z) \, dz 
- \overline{F}_e(x + y) \psi(u) + \overline{F}_e(u + y) \right\},
\]

where

\[
C_{k,n}(u, x, y) : = \overline{F}_e(u + y - k d_n) - \overline{F}_e(u + y - (k - 1) d_n)
- d_n p_e(u + y - (k - 1) d_n) \geq 0 .
\]
For appropriate values of $x$ and $y$, the above three theorems yield the following bounds for $\psi(u)$.

**Proposition 3.** Let $U(u)$ be a function such that $\psi(u) \leq U(u)$ for all $u > 0$. Then for all such $u$, it holds that

$$
\psi(u) \leq \phi \frac{n}{u} \sum_{k=1}^{n} \left( \mathbb{P}(u - k \frac{u}{n}) - \mathbb{P}(u - (k - 1) \frac{u}{n}) \right)
\times \int_{(k-1) \frac{u}{n}}^{k \frac{u}{n}} U(z) \, dz + \phi \mathbb{P}(u).
$$

**Proposition 4.** Let $L(u)$ be such that $\psi(u) \geq L(u)$ for $u > 0$. Then for any $n = 1, 2, \ldots$, it holds that

$$
\psi(u) \geq \phi \sum_{k=1}^{n+1} p_e \left( u - (k - 1) \frac{u}{n} \right) \int_{z_{k-1,n}}^{z_{k,n}} L(z) \, dH(z) + \phi \mathbb{P}(u).
$$

**Proposition 5.** Let $L(u)$ be such that $\psi(u) \geq L(u)$ for $u > 0$. Then for any $n = 1, 2, \ldots$, it follows that

$$
\psi(u) \geq \phi \sum_{k=1}^{n} p_e \left( u - (k - 1) \frac{u}{n} \right) \int_{(k-1) \frac{u}{n}}^{k \frac{u}{n}} L(z) \, dz + \phi \sum_{k=1}^{n} C_{k,n}(u) L(k \frac{u}{n}) + \phi \mathbb{P}(u) \frac{1}{1 - \phi C_{n,n}(u)}.
$$

Our result are illustrated by several examples even in the case where $\psi$ is not known.

**References**


