ABSTRACT

The techniques presented in this paper are applicable to the valuation of general corporate liabilities, corporate loans etc. However, merely for presentation purposes we choose to limit the discussion to defaultable corporate bonds. It is important to note though that most of the present results are of interest in their own right in the theory of semi-Markov processes. Credit ratings of defaultable corporate bonds are typically identified with elements of a finite set, referred to as the set of credit classes or credit grades. The classical way of modeling the evolution of the credit migrations is in terms of either a discrete-or continuous-time homogeneous Markov chain, first introduced by Jarrow and Turnbull(1995) and Jarrow et al.(1997). Works related to the simple Markov chain approach are those by Kijima(1998), Kijima and Komoribayashi(1998), Lando(1998),(2000), and Duffie and Singleton(1999). An excellent review of the above works, the mathematical foundation of their models, and the determination of their common core and interrelations are given in Bielecki and Rutkowski(2002). However in the study of Carty and Fons(1994), it was established that the duration of stay in a specific credit rating class followed the Weibull distribution. The inhomogeneity in time of transition probabilities has been reported by many authors, among whom are Duffie and Singleton(2003), Lando and Skotheberg(2001), Hamilton(2001) and Jonsson and Frison(1996). Recently in Vasileiou and Vassiliou(2006) the evolution of the credit migration of a corporate bond was modeled as an inhomogeneous semi-Markov chain.

Let the real-world filtered probability space ($\Omega, \mathcal{G}, Q$) where $Q$ is interpreted as a real world probability measure, and let the filtration $\mathcal{G}_t \subseteq \mathcal{G}$, $t = 0, 1, 2, \ldots$, model the flow of all observations available to traders. Let $T^*$ be a fixed horizon date. We assume a family $B(t, T)$, $t \leq T \leq T^*$ of adapted processes to be an arbitrage free family (Shreve(2004)). Let $Q^*$ be the martingale measure for the family $B(t, T)$ (Shreve(2004), Musiela and Rutkowski(1997)) and let $Q_T$ be the forward martingale measure. Also let the Radon-Nikodym derivative be given by

$$
\frac{dQ^*}{dQ} \bigg|_{\mathcal{G}_T} = \psi_T \quad Q-a.s.
$$

where the $\mathcal{G}_T$-measurable random variable $\psi_T$ is strictly positive $Q-a.s.$ and $\mathbb{E}_Q(\psi_T) = 1$. Then the density process $\psi_t = \mathbb{E}_Q(\psi_T | \mathcal{G}_t)$, $t = 0, 1, \ldots, T$ follows a strictly positive martingale under $Q$. Now it is found in Vasileiou and Vassiliou(2006) that the real-world probability measure $Q$ and the forward probability measure are equivalent on $(\Omega, \mathcal{G}_T)$, and that their Randon-Nikodym derivative will be given by

$$
\frac{dQ_T}{dQ} \bigg|_{\mathcal{G}_T} = \frac{\psi_T}{B_T B(0, T)} = \theta_T \quad Q-a.s. \quad \text{and} \quad B_T = \prod_{u=0}^{t} (1 + r_u)
$$

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where \( r_t \) denotes the interest rate process for \( t = 0, 1, 2, \ldots \). We assume that the evolution of a defaultable bond taking values in \( K = \{1, 2, \ldots, k, k + 1 \} \) follows a discrete-time \( G \)-inhomogeneous semi-Markov process \( \{X_t, S_{t+}\}_{t=0}^\infty \) as is given in definition 3.1 in Vasileiou and Vassiliou (2006). Let \( p_{ij}(t) = \mathbb{Q}\{S_{t+} = j \mid X_t = i\} \), \( \varpi_{ij}(t) \) the duration of a bond which entered in grade \( i \) at time \( t \) governed by the probability density function \( h_{ij}(t, m) = \mathbb{Q}\{\varpi_{ij}(t) = m \mid S_{t+} = j, X_t = i\} \). In Vasileiou and Vassiliou (2006) the following theorem was proved:

**Theorem 0.1.** Let the random variable \( \theta^{-1}_t \theta_{t+} \) be \((\sigma(X_t) \lor \sigma(S_{t+})) \)-measurable for any \( t=0,1,\ldots,T-1 \) i.e. for every such \( t \) we have \( \theta^{-1}_t \theta_{t+} = g_t(X_t, S_{t+}) \) for some function \( g_t : K \times K \rightarrow \mathbb{R} \). Also assume that \( \theta^{-1}_t \theta_{t+} \) is \((\sigma(X_t) \lor \sigma(S_{t+}) \lor \sigma(\varpi_{X_t, S_{t+}}(t))) \)-measurable for any \( t=0,1,\ldots,T-1 \), in other words \( \theta^{-1}_t \theta_{t+} = f_{t+}(X_t, S_{t+}, \varpi_{X_t, S_{t+}}(t)) \) for some function \( f_{t+} : K \times K \times \mathbb{R} \rightarrow \mathbb{R} \). Then the pair \( \{X_t, S_{t+}\}_{t=0}^\infty \) also follows a discrete time \( G \)-inhomogeneous semi-Markov process under \( \mathbb{Q}_T \) and, in addition, a) \( \mathbb{Q}_T \{S_{t+} = j \mid X_t = i\} = g_t(i, j, p_{ij}(t)) \) and b) \( h_{ij}(t, m) = \mathbb{Q}_T \{\varpi_{ij}(t) = m \mid S_{t+} = j, X_t = i\} = f_{t+}(i, j, m, h_{ij}(t, m)) \).

In the present we replace the assumption of theorem 01 with the following assumptions that in the special case of a homogeneous Markov chain coincide with the one made by Bielecki and Rutkowski (2002): a) There exist a finite set \( A \) such that for any \( t=0,1,\ldots,T-1 \), \( \theta^{-1}_t \theta_{t+} = 1 + \sum_{a \in A} g_t^a(X_t) g_{t+}^a(X_t, S_{t+}) \). b) There exist a finite set \( B \) such that \( \theta^{-1}_t \theta_{t+} = 1 + \sum_{b \in B} f_{t+}^b(X_t, S_{t+}) f_{t+}^b(X_t, S_{t+}, \varpi_{X_t, S_{t+}}(t)) \) for any \( t=0,1,\ldots,T-1 \). An immediate consequence of the new assumptions is that the transition probabilities sequences and the corresponding real world probability measure are martingales greater than \(-1\) for every \( t \) and we find a more suitable form for our purposes. Finally we prove a basic theorem where we provide simplified closed analytic functional relationships between the forward probability measure transition probabilities sequences and the corresponding real world probability measure ones. These relationships provide the means for a general calibration method of the proposed inhomogeneous semi-Markov model for the evolution of a defaultable bond.

**REFERENCES**


