

# Proving the Regularity of the Minimal Probability of Ruin via a Game of Stopping and Control

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joint work with  
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- ▶ Assume that  $B^c$  and  $B^S$  are correlated Brownian motions with correlation coefficient  $\rho \in [-1, 1]$ .
- ▶ The wealth dynamics

$$dW_t = (r W_t + (\mu - r) \pi_t - c_t) dt + \sigma \pi_t dB_t, \quad W_0 = w > 0.$$

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- ▶  $\tau_0 = \inf\{t \geq 0 : W_t \leq 0\}$ .
- ▶  $\tau_d$  is exponentially distributed with parameter  $\lambda$  (Time of death).

## Our Goal

$\psi$  given is decreasing and convex with respect to  $w$ , increasing with respect to  $c$  and is the unique classical solution of the following HJB equation

$$\begin{aligned} \lambda v &= (rw - c) v_w + a c v_c + \frac{1}{2} b^2 c^2 v_{cc} \\ &+ \min_{\pi} \left[ (\mu - r) \pi v_w + \frac{1}{2} \sigma^2 \pi^2 v_{ww} + \sigma \pi b c \rho v_{wc} \right], \quad (1) \\ v(0, c) &= 1 \text{ and } v(w, 0) = 0. \end{aligned}$$

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$\psi$  given is decreasing and convex with respect to  $w$ , increasing with respect to  $c$  and is the unique classical solution of the following HJB equation

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The optimal investment strategy  $\pi^*$  is given in feedback form by

$$\pi_t^* = - \frac{(\mu - r) \psi_w(W_t^*, c_t) + \sigma b \rho c_t \psi_{wc}(W_t^*, c_t)}{\sigma^2 \psi_{ww}(W_t^*, c_t)},$$

in which  $W^*$  is the optimally controlled wealth process.

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- ▶ **Analysis of the Controller and Stopper Problem.**

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It turns out that  $\psi(w, c) = \phi(w/c)$  in which  $\phi$  is the unique classical solution of the following HJB equation on  $\mathbb{R}_+$  :

$$\lambda f = (\tilde{r}z - 1) f' + \frac{1}{2} b^2 (1 - \rho^2) z^2 f'' + \min_{\tilde{\pi}} \left[ (\mu - r - \sigma b \rho) \tilde{\pi} f' + \frac{1}{2} \sigma^2 \tilde{\pi}^2 f'' \right], \quad (2)$$
$$f(0) = 1 \text{ and } \lim_{z \rightarrow \infty} f(z) = 0,$$

in which  $\tilde{r} = r - a + b^2 + (\mu - r - \sigma b \rho) \rho b / \sigma$ .

# Interpretation of the Reduced Problem

Consider two (risky) assets with prices  $\tilde{S}^{(1)}$  and  $\tilde{S}^{(2)}$  following the diffusions

$$d\tilde{S}_t^{(1)} = \tilde{S}_t^{(1)} \left( \tilde{r} dt + b\sqrt{1 - \rho^2} d\tilde{B}_t^{(1)} \right),$$

and

$$d\tilde{S}_t^{(2)} = \tilde{S}_t^{(2)} \left( \tilde{\mu} dt + \sqrt{b^2(1 - \rho^2) + \sigma^2} d\tilde{B}_t^{(2)} \right),$$

in which  $\tilde{\mu} = \mu - r + \sigma b\rho + \tilde{r}$ .

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in which  $\tilde{\mu} = \mu - r + \sigma b\rho + \tilde{r}$ . Also,  $\tilde{B}^{(1)}$  and  $\tilde{B}^{(2)}$  are correlated standard Brownian motions with correlation coefficient

$$\tilde{\rho} = \frac{b\sqrt{1 - \rho^2}}{\sqrt{b^2(1 - \rho^2) + \sigma^2}}.$$

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- ▶ With a slight abuse of notation, let  $\tilde{\pi}_t$  be the dollar amount that the individual invests in the second asset at time  $t$ ; then,  $Z_t - \tilde{\pi}_t$  is the amount invested in the first asset at time  $t$ .

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- ▶ The function  $\phi$  is again a minimum probability of lifetime ruin!

$$\phi(z) = \inf_{\tilde{\pi} \in \tilde{\mathcal{A}}} \tilde{\mathbb{P}}^z (\tilde{\tau}_0 < \tau_d)$$

# An Approximating Sequence

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- ▶ Let us define the auxiliary problem

$$\phi_M(z) = \inf_{\tilde{\pi} \in \tilde{\mathcal{A}}} \tilde{\mathbb{P}}^z (\tilde{\tau}_0 < (\tilde{\tau}_M \wedge \tau_d)),$$

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$$\lambda f = (\tilde{r}z - 1) f' + \frac{1}{2} b^2 (1 - \rho^2) z^2 f'' +$$
$$\min_{\tilde{\pi}} \left[ (\mu - r - \sigma b \rho) \tilde{\pi} f' + \frac{1}{2} \sigma^2 \tilde{\pi}^2 f'' \right],$$
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$$f(0) = 1, \quad f(M) = 0.$$

- ▶ Furthermore, on  $\mathbb{R}_+$ , we have

$$\lim_{M \rightarrow \infty} \phi_M(z) = \phi(z).$$



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Define a controlled stochastic process  $Y^\alpha$  by

$$dY_t^\alpha = Y_t^\alpha \left[ (\lambda - \tilde{r}) dt + \frac{\mu - r - \sigma b \rho}{\sigma} d\hat{B}_t^{(1)} \right] + \alpha_t \left[ b\sqrt{1 - \rho^2} dt + d\hat{B}_t^{(2)} \right].$$

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Admissible strategies,  $\mathcal{A}(y)$ :  $(\alpha_t)_{t \geq 0}$  that satisfy the integrability condition  $\mathbb{E}[\int_0^t \alpha_s^2 ds] < \infty$ , and  $Y_t^\alpha \geq 0$  almost surely, for all  $t \geq 0$ .

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The controller-and-stopper problem

$$\hat{\phi}_M(y) = \sup_{\alpha \in \mathcal{A}(y)} \inf_{\tau} \hat{\mathbb{E}}^y \left[ \int_0^{\tau} e^{-\lambda t} Y_t^{\alpha} dt + e^{-\lambda \tau} u_M(Y_{\tau}^{\alpha}) \right],$$

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Here “payoff function”  $u_M$  for  $y \geq 0$  is given by

$$u_M(y) := \min(My, 1).$$

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  - ▶ Suppose that  $y_1 > 0$  is such that  $\hat{\phi}_M(y_1) = u_M(y_1)$ . First, suppose that  $y_1 \leq 1/M$ ; then, because  $\hat{\phi}_M(0) = 0$  and because  $\hat{\phi}_M$  is non-decreasing, concave, and bounded above by the line  $My$  it must be that  $\hat{\phi}_M(y) = My$  for all  $0 \leq y \leq y_1$ . Thus, if  $y_1 \leq 1/M$  is not in  $D$ , then the same is true for  $y \in [0, y_1]$ .

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  - ▶ Suppose that  $y_1 > 0$  is such that  $\hat{\phi}_M(y_1) = u_M(y_1)$ . First, suppose that  $y_1 \leq 1/M$ ; then, because  $\hat{\phi}_M(0) = 0$  and because  $\hat{\phi}_M$  is non-decreasing, concave, and bounded above by the line  $My$  it must be that  $\hat{\phi}_M(y) = My$  for all  $0 \leq y \leq y_1$ . Thus, if  $y_1 \leq 1/M$  is not in  $D$ , then the same is true for  $y \in [0, y_1]$ .
  - ▶ Finally, suppose that  $y_1 \geq 1/M$ ; then, because  $\hat{\phi}_M$  is non-decreasing, concave, and bounded above by the horizontal line 1 it must be that  $\hat{\phi}_M(y) = 1$  for all  $y \geq y_1$ . Thus, if  $y_1 \geq 1/M$  is not in  $D$ , then the same is true for  $y \in [y_1, \infty)$ .

# Viscosity Solutions

$g \in C^0(\mathbb{R}_+)$  is a viscosity supersolution (respectively, subsolution) if

$$\max \left[ \lambda g(y_1) - y_1 - (\lambda - \tilde{r})y_1 f'(y_1) - m y_1^2 f''(y_1) \right. \\ \left. - \max_{\alpha} \left[ b \sqrt{1 - \rho^2} \alpha f'(y_1) + \frac{1}{2} \alpha^2 f''(y_1) \right], \right. \\ \left. g(y_1) - u_M(y_1) \right] \geq 0$$

(respectively,  $\leq 0$ ) whenever  $f \in C^2(\mathbb{R}_+)$  and  $g - f$  has a global minimum (respectively, maximum) at  $y = y_1 \geq 0$ . (ii)  $g$  is a viscosity solution of if it is both a viscosity super- and subsolution.

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- ▶ Suppose  $M > 1/\lambda$ , and suppose that  $D$  is empty. Then, for all  $y \geq 0$ , we have  $\hat{\phi}_M(y) = u_M(y) = \min(My, 1)$ .  $\hat{\phi}_M = u_M$  is a viscosity solution. Because  $M > 1/\lambda$ , there exists  $y_1 \in (1/M, \lambda)$ . The value function is identically 1 in a neighborhood of  $y_1$ , the QVI evaluated at  $y = y_1$  becomes  $\max[\lambda - y_1, 0] = 0$ , which contradicts  $y_1 < \lambda$ . Thus, the region  $D$  is non-empty.

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Assume that

$$D_+ \hat{\phi}_M(y_0) < D_- \hat{\phi}_M(y_0).$$

Let

$$\delta \in (D_+(y_0)\hat{\phi}_M, D_-\hat{\phi}_M(y_0)).$$

. Then the function

$$\psi_\varepsilon(y) = 1 + \delta(y - y_0) - \frac{(y - y_0)^2}{2\varepsilon},$$

dominates  $\hat{\phi}_M$  locally at  $y_0$ . Since  $\hat{\phi}_M$  is a viscosity subsolution of we have that

$$\lambda - y_0 - (\lambda - \tilde{r})\lambda\delta + \frac{m\lambda^2}{\varepsilon} + \frac{1}{2}b^2(1 - \rho^2)\frac{\delta^2}{\varepsilon} \leq 0.$$

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$$\lambda g = y + (\lambda - \tilde{r})yg' + my^2g'' + \max_{\alpha} \left[ b\sqrt{1 - \rho^2}\alpha g' + \frac{1}{2}\alpha^2 g'' \right] \quad \text{on } D,$$

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$$g(y_M) = My_M \text{ and } g(y_0) = 1.$$

- ▶ The value function for this problem, namely  $\hat{\phi}_M$ , is non-decreasing (strictly increasing on  $D$ ), concave (strictly concave on  $D$ ), and  $C^2$  on  $\mathbb{R}_+$  (except for possibly at  $y_M$  where it is  $C^1$ ).

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- ▶ If  $z \geq M$ , then  $\Phi_M(z) = 0$  because  $\hat{\phi}_M(y) \leq u_M(y) \leq My \leq zy$ , from which it follows that the maximum on the right-hand side of (\*\*) is achieved at  $y^* = y_M$ .

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- ▶ When  $z < M$ ,  $y^* = I_M(z)$  maximizes (\*\*), in which  $I_M$  is the inverse of  $\hat{\phi}'_M$  on  $(y_M, y_0]$ .

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- ▶ Which implies

$$\begin{aligned}\Phi'_M(z) &= \hat{\phi}'_M [I_M(z)] I'_M(z) - I_M(z) - zI'_M(z) \\ &= zI'_M(z) - I_M(z) - zI'_M(z) = -I_M(z).\end{aligned}$$

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- ▶ Taking one more derivative

$$\Phi''_M(z) = -I'_M(z) = -1/\hat{\phi}''_M [I_M(z)].$$

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Rewrite this equation in terms of  $\Phi_M$  to get

$$\lambda \Phi_M(z) = (\tilde{r}z - 1) \Phi'_M(z) - m \frac{(\Phi'_M(z))^2}{\Phi''_M(z)} + \frac{1}{2} b^2 (1 - \rho^2) z^2 \Phi''_M(z).$$

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$$\begin{aligned}\lambda \hat{\phi}_M [I_M(z)] &= I_M(z) + (\lambda - \tilde{r}) I_M(z) \hat{\phi}'_M [I_M(z)] + m I_M^2(z) \hat{\phi}''_M [I_M(z)] \\ &\quad - \frac{1}{2} b^2 (1 - \rho^2) \frac{(\hat{\phi}'_M [I_M(z)])^2}{\hat{\phi}''_M [I_M(z)]}.\end{aligned}$$

Rewrite this equation in terms of  $\Phi_M$  to get

$$\lambda \Phi_M(z) = (\tilde{r}z - 1) \Phi'_M(z) - m \frac{(\Phi'_M(z))^2}{\Phi''_M(z)} + \frac{1}{2} b^2 (1 - \rho^2) z^2 \Phi''_M(z).$$

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Thanks to a verification theorem  $\Phi_M = \phi_M$ .

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- ▶ A verification theorem shows that  $\psi(w, c) = \phi(w/c)$ .

# References

# References

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Thanks for your attention!