The Strong Law of Large Numbers for Extended Negatively Dependent Random Variables

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Definition

- LEND

  $\{X_k, k = 1, \ldots, n\}$ are said to be LEND if there is some $M > 0$ such that, for all x_k, $k = 1, \ldots, n$,

$$\Pr\left(\bigcap_{k=1}^{n} (X_k \leq x_k)\right) \leq M \prod_{k=1}^{n} \Pr(X_k \leq x_k); \qquad (1)$$

- UEND

  if

$$\Pr\left(\bigcap_{k=1}^{n} (X_k > x_k)\right) \leq M \prod_{k=1}^{n} \Pr(X_k > x_k); \qquad (2)$$

- END

  if they are both LEND and UEND.

Comments on END

- When $M = 1$, inequalities (1) and (2) describe lower and upper negative dependence, respectively.
- Ebrahimi and Ghosh (1981) and Block et al. (1982): first proposed the concept of negative dependence, which has been extensively investigated since then.
- The concept of END was proposed by Liu (2009) and further promoted by Chen et al. (2010) in the study of precise large deviations.
- The END structure covers all negative dependence structures and, more interestingly, it covers certain positive dependence structures.

Sufficient Condition for LEND/UEND

Lemma 1 Assume that random variables X_k, $k = 1, \ldots, n$, have continuous marginal distributions, respectively.

(a) If every marginal copula density is bounded in a neighborhood of the origin (whose coordinates are all 0), then X_k, $k = 1, \ldots, n$, are LEND;

(b) If every marginal copula density is bounded in a neighborhood of the ultimate vertex (whose coordinates are all 1), then X_k, $k = 1, \ldots, n$, are UEND.

Farlie-Gumbel-Morgenstern (FGM) Distribution

The n-dimensional Farlie-Gumbel-Morgenstern (FGM) distribution is given by

$$F_{1,\dots,n}(x_1,\dots,x_n) = \left(\prod_{k=1}^{n} F_k(x_k)\right)\left(1 + \sum_{1\le i<j\le n} a_{ij}\overline{F_i}(x_i)\overline{F_j}(x_j)\right),$$

where $F_k = 1 - \overline{F_k}$, $k = 1, \dots, n$, are corresponding marginal distributions and a_{ij} are real numbers chosen such that $F_{1,\dots,n}$ is a proper n-dimensional distribution.

By Lemma 1, every n-dimensional FGM distribution describes a specific END structure.

Motiviation

- Matuła (1992): established the strong law of large numbers for pairwise negatively dependent random variables. A key step of Matuła's (1992) derivation is that, by Hoeffding's identity, the covariance of two negatively dependent random variables, being suitably truncated, is non-positive. Hence, the pairwise negative dependence greatly prevents the partial sums from diverging to infinite.
- However, in general, this implication is not true for END random variables.
- Recent developments of the strong law of large numbers for negatively dependent random variables can be found in Bingham and Nili Sani (2004), Gerasimov (2009), and Baek et al. (2009), among others.

Main Result 1

Theorem 1 Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of END random variables with common distribution F. Denote by S_n its nth partial sum, $n = 1, 2, \ldots$. Then $S_n/n \overset{\text{a.s.}}{\to} \mu$ as $n \to \infty$ for some real number μ if and only if $\mathrm{E}|X_1| < \infty$, and for each case $\mu = \mathrm{E}X_1$.

Lemma 2

Let $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$ be the positive and negative parts of real number x, respectively.

Lemma 2 For random variables X_k and real functions g_k,

(a) If $\{X_k\}$ are UEND with some dominating coefficient $M > 0$, then

$$\mathrm{E}\left(\prod_{k=1}^{n} X_k^+\right) \leq M \prod_{k=1}^{n} \mathrm{E} X_k^+.$$

(b) Assume that $\{X_k\}$ are LEND/UEND/END with some dominating constant $M > 0$. If g_k are all non-decreasing then $g_k(X_k)$ are still LEND/UEND/END, while if g_k are all non-increasing then $g_k(X_k)$ are UEND/LEND/END. For each case, the dominating constant $M > 0$ remains unchanged.

Generalized Borel-Cantelli Lemma

The following generalized Borel-Cantelli lemma is due to Kochen and Stone (1964) and was retrieved recently by Yan (2006):

Lemma 3 Let $\{A_n, n = 1, 2, \ldots\}$ be a sequence of events such that $\sum_{n=1}^{\infty} \Pr(A_n) = \infty$. Then

$$\Pr(A_n \text{ i.o.}) \geq \limsup_{n\to\infty} \frac{\sum\limits_{1\leq i<j\leq n} \Pr(A_i)\Pr(A_j)}{\sum\limits_{1\leq i<j\leq n} \Pr(A_i A_j)}.$$

Auxiliary Functions

Let F be a distribution on $(-\infty, \infty)$. For arbitrarily fixed $\delta > 0$, define auxiliary functions f_δ, $f_\delta^{\pm}$ as

$$\begin{aligned} f_\delta(x) &= x^{-\delta} \int_{-x}^{x} |y|^{1+\delta} F(\mathrm{d}y) \\ &= x^{-\delta} \int_{0}^{x} y^{1+\delta} F(\mathrm{d}y) + x^{-\delta} \int_{-x}^{0} (-y)^{1+\delta} F(\mathrm{d}y) \\ &= f_\delta^{+}(x) + f_\delta^{-}(x), \qquad x > 0. \end{aligned} \tag{3}$$

These auxiliary functions will be crucial for establishing our key inequalities for the tail probabilities of the sums of END random variables.

Lemma 4

Lemma 4 For the auxiliary functions f_δ, $f_\delta^{\pm}$ defined in (3), as $x \to \infty$,

(a) if $x\overline{F}(x) \to 0$ then $f_\delta^{+}(x) \to 0$;

(b) if $xF(-x) \to 0$ then $f_\delta^{-}(x) \to 0$;

(c) if $x\left(\overline{F}(x) + F(-x)\right) \to 0$ then $f_\delta(x) = f_\delta^{+}(x) + f_\delta^{-}(x) \to 0$.

Truncation

Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of random variables with common distribution F and mean zero. For arbitrarily fixed $0 < v < 1$, define

$$\widetilde{X}_k = -vx1_{(X_k \le -vx)} + X_k 1_{(-vx < X_k \le vx)} + vx1_{(X_k > vx)}. \qquad (4)$$

Write

$$\widetilde{S}_n = \sum_{k=1}^{n} \widetilde{X}_k, \qquad n = 1, 2, \ldots,$$

and $\mu_\pm = \mathrm{E}X_1^\pm$. Trivially, $\mu_+ = \mu_-$ if $\mathrm{E}X_1 = 0$.

Lemma 5

Lemma 5 Consider the truncated random variables defined in (4), where X_k, $k = 1, 2, \ldots$, are END random variables with common distribution F, mean zero, and a dominating constant $M > 0$. Then for every $v > 0$, $\gamma > 0$, $0 < \delta \leq 1$, and $0 < \theta < 1$, there is some $x_0 = x_0(v, \gamma, \delta, \theta) > 0$ such that, for all $n = 1, 2, \ldots$ and $x \geq (\gamma n) \vee x_0$,

$$\Pr\left(\left|\widetilde{S}_n\right| > x\right) \leq 2M\left(f_\delta(vx) + vx\Pr\left(|X_1| > vx\right)\right)^{(1-\theta)/v},$$

where the auxiliary function f_δ is defined in (3).

Lemma 6

Lemma 6 Recall the truncated random variables defined in (4). In addition to the conditions of Lemma 5, assume that $\mathrm{E}\left|X_1\right|^{1+\delta} < \infty$ for some $0 < \delta \leq 1$, then, for every $v > 0$, $\gamma > 0$, and $0 < \theta < 1$, there is some $K = K(v, \gamma, \delta, \theta) > 0$ such that, for all $n = 1, 2, \ldots$ and $x \geq \gamma n$,

$$\Pr\left(\left|\widetilde{S}_n\right| > x\right) \leq Kx^{-\delta(1-\theta)/v}.$$

A Random Sum

Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of random variables, and let N be a nonnegative integer-valued random variable independent of $\{X_k, k = 1, 2, \ldots\}$. The study of the tail behavior of the random sum

$$S_N = \sum_{k=1}^{N} X_k \tag{5}$$

is of fundamental interest in various areas of applied probability.

Robert and Segers (2008) interpreted S_N as the total amount of claims of an insurance portfolio in earthquake insurance.

Consistent Variation

A distribution G on $[0, \infty)$ is said to be of consistent variation, written as $G \in \mathcal{C}$, if

$$\lim_{y \to 1-} \limsup_{x \to \infty} \frac{\overline{G}(xy)}{\overline{G}(x)} = 1$$

Note that the class $\mathcal{C}$ contains all distributions of regular variation.

In Foss' talk on 3rd June, the class $\mathcal{C}$ was named as the class $\mathcal{IRV}$.

Literature Review

Assuming that N has a consistently-varying tail and that X_k, $k = 1, 2, \ldots$, are independent, identically distributed, and nonnegative, with tails relatively lighter than that of N, Robert and Segers (2008) showed that the tail behavior of S_N is mainly determined by that of N. See also Aleškevičienė et al. (2008) and Denisov et al. (2009) for some extensions.

Main Result 2

Under the help of Theorem 1 and Lemma 6, we are able to relax the independence assumption on $\{X_k, k = 1, 2, \ldots\}$ to END and real valued.

Theorem 2 Consider the random sum (5) in which $\{X_k, k = 1, 2, \ldots\}$ is a sequence of END random variables with common distribution F, mean $\mu > 0$, and $\mathrm{E}\,|X_1|^{1+\delta} < \infty$ for some $\delta > 0$, while N follows a distribution $G \in \mathcal{C}$. As $x \to \infty$, the relation

$$\Pr(S_N > x) \sim \overline{G}(x/\mu)$$

holds under one of the following groups of conditions:

(a) $x \Pr(|X_1| > x) = o(\overline{G}(x))$;

(b) $\mathrm{E}N < \infty$ and $\Pr(|X_1| > x) = o(\overline{G}(x))$.

A Quasi-renewal Counting Process

Let $\{N_t, t \geq 0\}$ be a quasi-renewal counting process defined as

$$N_t = \max\left\{ n = 1, 2, \ldots : \sum_{k=1}^{n} Y_k \leq t \right\}, \qquad t \geq 0, \tag{6}$$

where the inter-arrival times Y_k, $k = 1, 2, \ldots$, form a sequence of nonnegative, END, and identically distributed random variables with common distribution G and finite, positive mean $1/\lambda$.

Main Result 3

Theorem 3 Consider the quasi-renewal counting process defined by (6). As $t \to \infty$,

(a) $N_t / (\lambda t) \stackrel{\text{a.s.}}{\to} 1$;

(b) $\mathrm{E} N_t^p \sim (\lambda t)^p$ for every $p > 0$.

For the standard renewal counting process, (a) is well-known, see Asmussen (2003). (b) is new even for the standard renewal counting process.

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Thank You!