Risk measures in ordered normed linear spaces with non-empty cone interior

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Partially ordered linear spaces and risk measures

We consider the commodity-price duality $\langle E, E^* \rangle$ where E is a normed linear space containing all the financial positions and E^* is its norm-dual. An element f of E^* denotes a price in the sense that the price paid today by some investor in order to buy the position whose payoff is $x \in E$ tomorrow is $f(x) =< x, f >$. The partially ordered linear spaces are seminally present in risk measures theory, since in the case of $E = \mathbb{R}^\Omega$ with Ω : finite, where the axioms about the *coherent acceptance sets* $\mathcal{A} \subseteq \mathbb{R}^\Omega$ are introduced, two of these axioms resemble the sets introducing the partial ordering relations in linear spaces (see **Artzner et al. (1999)**):

$$\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}, \lambda\mathcal{A} \subseteq \mathcal{A}, \lambda \in \mathbb{R}_+.$$

Such sets are called *wedges* and if moreover $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$ are called *cones*. In general, the results of this work enforce the partially ordered linear spaces' point of view in risk measure theory, developed in **Jaschke and Küchler (2001)**, **Fritelli and Gianin (2002)** for infinite -dimensional spaces.

This point of view refers to a triplet

$$(\mathcal{A}, E_+, e)$$

called **insurance triplet** where:

- E is a normed linear space, mostly a space of random variables (e.g. $L^p(\Omega, \mathcal{F}, P)$)
- $\mathcal{A} \subseteq E$ is the *acceptance set* used by the regulator,
- $E_+ \subseteq \mathcal{A}$ is a wedge (cone) of E inducing a partial ordering relation $\geq$ on E which indicates 'the less and the more' on E.
- E_+ may be taken as the intersection of the acceptance sets $\mathcal{A}_i, i \in I$, for a set of investors $I \neq \emptyset$. E_+ contains all the 'commonly good' investments
- The asset $e \in E_+$ is the 'insurance instrument' used, or else the element such that for any $x \in E$ the minimum number of shares $\alpha \in \mathbb{R}$ are determined such that $x + \alpha e \in \mathcal{A}$.

Through this work we emphasize on:

(a) **the geometric properties of the 'insurance instrument' asset**. In most of the results, E is reflexive, the cone E_+ has norm-interior points and $e \in intE_+$. This case has been studied for non-reflexive spaces in the case of L^∞_+ (see in **Delbaen (2002)**).

(b) **the properties of the partial ordering $\geq$ induced by the cone E_+**. By the previous assumptions, we may move beyond the case of Banach lattices which are well-known cases of partially ordered spaces in which continuity of risk measures is assured (see **Biagini and Fritelli (2009)**). We prove continuity for partially ordered spaces whose ordering cones are such that the ordering relation is a non-lattice one. Such a cone is a Bishop-Phelps cone in an infinite-dimensional normed space E

$$K(f,a) = \{x \in E | f(x) \geq a\|x\|\}, a \in (0,1), f \in E^*, \|f\| = 1. \tag{1}$$

(c) **we prove dual representation results where the representation variable is the spot price functional π faced by the investors in the market.** This kind of representation for convex risk measures extends the relevant Th.2 of **Jaschke and Küchler (2001)**. Also, indicated cases of coherent risk measures are such the set of representing functionals of a coherent risk measure is bounded such as in the case where the state space is finite.

(d) **the fact that for these cases, we prove the Lipschitz continuity of the relevant convex risk measures**. This fact implies all the other forms of continuity (e.g. Fatou continuity) mentioned in literature about risk measures.

Coherent and Convex risk measures

Suppose that E is a partially ordered (reflexive) Banach space whose ordering cone is E_+. We suppose that the norm-interior of E_+ is non-empty and consider e to be such a point. Then a function $\rho : E \to \mathbb{R}$ is called e-**convex** risk measure if it satisfies the following properties:

(i) $\rho(x + \alpha e) = \rho(x) - \alpha$ for all $x \in E, \alpha \in \mathbb{R}$ (e-Translation Invariance)

(ii) $\rho(\lambda x + (1 - \lambda)y) \leq \lambda\rho(x) + (1 - \lambda)\rho(y)$ for all $x, y \in E, \lambda \in [0, 1]$ (Convexity)

(iii) $\rho(x) \geq \rho(y)$ if $y \geq x$ with respect to the partial ordering whose positive cone is E_+ (E_+-Monotonicity)

If ρ also satisfies the Positive Homogeneity Property, where $\rho(\lambda x) = \lambda\rho(x)$ for all $x \in E$ and $\lambda \in \mathbb{R}_+$, then ρ is called e-**coherent**.

Dual representation results

Theorem 1. *If $\rho : E \to \mathbb{R}$ is an e-coherent risk measure whose acceptance set $\mathcal{A}_\rho$ is $\sigma(E, E^*)$- closed, then*

$$\rho(x) = \sup\{\pi(-x) \,|\, \pi \in B\},$$

for any $x \in E$, where $B = \{y \in \mathcal{A}_\rho^0 \,|\, \hat{e}(y) = 1\} = B_e \cap \mathcal{A}_\rho^0$.

Theorem 2. *If $\rho : E \to \mathbb{R}$ is an e-convex risk measure with $\sigma(E, E^*)$-closed acceptance set, then*

$$\rho(x) = \sup\{\pi(-x) - a(\pi) \,|\, \pi \in B_e\} \tag{2}$$

for any $x \in E$, where $B_e = \{y \in E_+^0 \,|\, \hat{e}(y) = 1\}$ and $a : B_e \to \overline{\mathbb{R}}$ is a 'penalty function' associated with ρ, with $a(\pi) \in (-\infty, \infty]$ for any $\pi \in B_e$. The e-convexity of every function ρ defined by 2 is obvious.

Remarks on the Duality results

- The theorem mainly used in the proofs is the Strong Separation Theorem for Convex Sets in locally convex spaces.

- The 'penalty function' $\alpha(\pi) = \sup\{\pi(-x) | x \in \mathcal{A}_\rho\}$ where $\mathcal{A}_\rho = \{x \in E | \rho(x) \leq 0\}$, is defined in a similar way with Th.5 in **Föllmer and Schied (2002)** for Euclidean spaces of positions.

- If $e = \mathbf{1}$ and $E = L^p(\Omega, \mathcal{F}, P), 1 < p < \infty$, then by considering the set of probability measures

$$\mathcal{Q}_{B_\mathbf{1}} = \{Q \in ca(\Omega) | Q << P, \frac{dQ}{dP} = \pi, \pi \in B_\mathbf{1}\},$$

the dual representation of every convex risk measure of the form described in Theorem 2 is

$$\rho(x) = \sup\{\mathbb{E}_Q(-x) - a(Q) \,|\, Q \in \mathcal{Q}_{B_\mathbf{1}}\}$$

such as in the Th. 5 of **Föllmer and Schied (2002)**.

- In the previous form, since the set of Radon-Nikodym derivatives $\frac{dQ}{dP}$ is bounded, the interiority assumption for $\mathbf{1}$ provides a case of finiteness and dual representation of convex risk measures on these spaces (see Th.2.11 in **Kaina and Rüschendorf (2009)**)

Continuity results

Proposition 1. *If E is a reflexive Banach lattice and $\rho : E \longrightarrow \mathbb{R}$ is an e- coherent risk measure with $\sigma(E, E^*)$-closed acceptance set $\mathcal{A}_\rho$, then ρ is a Lipschitz function.*

Proposition 2. *If E is reflexive and $\rho : E \longrightarrow \mathbb{R}$ is an e-convex (e-coherent) risk measure with $\sigma(E, E^*)$-closed acceptance set $\mathcal{A}_\rho$, then ρ is Lipschitz.*

Proposition 3. *If E is a reflexive L^p-space, $\rho : E \longrightarrow \mathbb{R}$ is either an e-coherent or an e-convex risk measure with $\sigma(E, E^*)$-closed acceptance set $\mathcal{A}_\rho$, then ρ is continuous from above, continuous from below, Fatou continuous and Lebesgue continuous.*

Definitions of Continuity

A risk functional $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ is called:

- *continuous from above* if for any $(x_n) \subseteq L^p, x_n \downarrow x,\ P$ a.s. with $x \in L^p$, this implies $lim_n \rho(x_n) = \rho(x)$.

- *continuous from below* if for any $(x_n) \subseteq L^p, |x_n| \leq y, x, y \in L^p, x_n \uparrow x,\ P$ a.s. with $x \in L^p$, this implies $lim_n \rho(x_n) = \rho(x)$.

- *Fatou continuous* if for any $(x_n) \subseteq L^p, |x_n| \leq y$, $x, y \in L^p, x_n \to x,\ P$ a.s., this implies $liminf_n \rho(x_n) \geq \rho(x)$.

- *Lebesgue continuous* if for any $(x_n) \subseteq L^p, |x_n| \leq y$, $x, y \in L^p, x_n \to x$, P a.s., this implies $lim_n \rho(x_n) = \rho(x)$.

About these definitions, see Def.3.1 in **Kaina and Rüschendorf (2009)**.

The Implications of Reflexivity

Suppose that E is a partially ordered normed linear space. By C^{00} we denote the subset $(C^0)^0$ of E^{**}. If C is a closed wedge of a reflexive space E, then $C^{00} = C$. Also, $f \in E^*$ is a uniformly monotonic functional of C if and only if $f \in intC^0$. If $E = K - K$ then the wedge K is called *generating*. If K is generating, then K^0 is a cone of E^*. If E is reflexive and e is an interior point of E, then E_+ is generating, hence E_+^0 is a cone. According to Th. 4 in **Polyrakis (2008)**, if e is an interior point of E_+ and $h \in E_+$ as a linear functional of E^* being a strictly positive linear functional of E_+^0, then the base

$$B_h = \{\pi \in E_+^0 \,|\, \hat{h}(\pi) = 1\}$$

defined by it is also bounded, hence weakly compact and B_e is so.

Other results

The norm $\|.\|$ of E is called *order-unit norm* with respect to e if the following equation holds for any $x \in E$

$$\|x\| = \inf\{\lambda > 0 | x \in [-\lambda e, \lambda e]\}.$$

(i) If $\rho : E \to \mathbb{R}$ is an e-coherent risk measure and $\|.\|$ is an order-unit norm with respect to e, E_+ is a $\sigma(E, E^*)$-closed cone and the acceptance set $\mathcal{A}_\rho$ is a cone, then ρ is Lipschitz. (In this case E is non -reflexive)

(ii) If C is a wedge of a normed linear space E which contains E_+ and $e \in intE_+$, then $\rho_C : E \to \overline{\mathbb{R}}$ is an e-coherent risk measure which does not take the value $+\infty$, where

$$\rho_C(x) = \inf\{\alpha \in \mathbb{R} | x + \alpha e \in C\}.$$

Further, if C is also a cone, then ρ_C does not take the value $-\infty$, too. (This result is contained in **Jaschke -Küchler (2001)** and it we give a detailed proof of it).

Partially ordered linear spaces

Let E be a (normed) linear space. A set $C \subseteq E$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbb{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. A pair $(E, \geq)$ where E is a linear space and $\geq$ is a binary relation on E satisfying the following properties:

(i) $x \geq x$ for any $x \in E$ (reflexive)

(ii) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in E$ (transitive)

(iii) If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbb{R}_+$ and $x + z \geq y + z$ for any $z \in E$, where $x, y \in E$ (compatible with the linear structure of E),

is called *partially ordered linear space*. The binary relation $\geq$ in this case is a *partial ordering* on E.

The set $P = \{x \in E | x \geq 0\}$ is called *(positive) wedge* of the partial ordering $\geq$ of E. Given a wedge C in E, the binary relation $\geq_C$ defined as follows:

$$x \geq_C y \iff x - y \in C,$$

is a partial ordering on E, called *partial ordering induced by C on E.*

If the partial ordering $\geq$ of the space E is *antisymmetric*, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in E$, then P is a cone. E' denotes the linear space of all linear functionals of E,while E^* is the norm dual of E^*, in case where E is a normed linear space. Suppose that C is a wedge of E. A functional $f \in E'$ is called *positive functional* of C if $f(x) \geq 0$ for any $x \in C$. $f \in E'$ is a *strictly positive functional* of C if $f(x) > 0$ for any $x \in C \setminus C \cap (-C)$. Some $f \in E'$ where E is a normed linear space, is called *uniformly monotonic functional* of C if there is some real number $a > 0$ such that $f(x) \geq a\|x\|$ for any $x \in C$.

In case where a uniformly monotonic functional of C exists, C is a cone. $C^0 = \{f \in E^* | f(x) \geq 0$ for any $x \in C\}$ is the *dual wedge of C in E^**. If C is a wedge of E^*, then the set $C_0 = \{x \in E | \hat{x}(f) \geq 0$ for any $f \in C\}$ is the *dual wedge of C in E*, where $\hat{}: E \to E^{**}$ denotes the natural embedding map from E into the second dual space E^{**} of E. Note that if for two wedges K, C of E $K \subseteq C$ holds, then $C^0 \subseteq K^0$. If C is a cone, then a set $B \subseteq C$ is called *base* of C if for any $x \in C \setminus \{0\}$ there exists a unique $\lambda_x > 0$ such that $\lambda_x x \in B$. The set $B_f = \{x \in C | f(x) = 1\}$ where f is a strictly positive functional of C is the *base of C defined by f*. B_f is bounded if and only if f is uniformly monotonic. If B is a bounded base of C such that $0 \notin \overline{B}$ then C is called *well-based*. If C is well-based, then a bounded base of C defined by a $g \in E^*$ exists. The Theorem 4 by Polyrakis (2008) we use is the following: Suppose that $\langle X, Y \rangle$ is a dual system. If X is a normed space, P a $\sigma(X, Y)$-closed cone of X so that the positive part $U_X^+ = U_X \cap P$ of the closed unit ball U_X of X is $\sigma(X, Y)$-compact, we have: either every base for P defined by a vector $y \in Y$ is bounded or every such base for P is unbounded.

A *lattice* is a partially ordered linear space E such that for any two elements $x, y \in E$ the supremum and the infimum of them with respect to the partial ordering of it exists. These elemetns are denoted by $x \vee y, x \wedge y$ respectively. If E is a linear lattice then for any $x \in E$ we may define the *positive part* $x^+ = x \vee 0$ of x, the *negative part* $x^- = (-x) \vee 0$ of x and the *absolute value* $|x| = x \vee (-x)$ of x. What holds in this case is $x = x^+ - x^-, |x| = x^+ + x^-$. A linear lattice E being also a normed space is called a *normed lattice* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ and if this normed space is a Banach space then it is called a *Banach lattice*.

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Thank you!