Risk measures in ordered normed linear spaces with non-empty cone interior

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Partially ordered linear spaces and risk measures

We consider the commodity-price duality $\langle E, E^* \rangle$ where $E$ is a normed linear space containing all the financial positions and $E^*$ is its norm-dual. An element $f$ of $E^*$ denotes a price in the sense that the price paid today by some investor in order to buy the position whose payoff is $x \in E$ tomorrow is $f(x) = \langle x, f \rangle$. The partially ordered linear spaces are seminally present in risk measures theory, since in the case of $E = \mathbb{R}^\Omega$ with $\Omega$ : finite, where the axioms about the coherent acceptance sets $A \subseteq \mathbb{R}^\Omega$ are introduced, two of these axioms resemble the sets introducing the partial ordering relations in linear spaces (see Artzner et al. (1999)):

$$A + A \subseteq A, \lambda A \subseteq A, \lambda \in \mathbb{R}_+.$$ 

Such sets are called wedges and if moreover $A \cap (-A) = \{0\}$ are called cones. In general, the results of this work enforce the partially ordered linear spaces' point of view in risk measure theory, developed in Jaschke and Küchler (2001), Fritelli and Gianin (2002) for infinite -dimensional spaces.
This point of view refers to a triplet 

\[(A, E_+, e)\]

called insurance triplet where:

- \(E\) is a normed linear space, mostly a space of random variables (e.g. \(L^p(\Omega, \mathcal{F}, P)\))
- \(A \subseteq E\) is the acceptance set used by the regulator,
- \(E_+ \subseteq A\) is a wedge (cone) of \(E\) inducing a partial ordering relation \(\geq\) on \(E\) which indicates 'the less and the more' on \(E\).
- \(E_+\) may be taken as the intersection of the acceptance sets \(A_i, i \in I\), for a set of investors \(I \neq \emptyset\). \(E_+\) contains all the 'commonly good' investments
- The asset \(e \in E_+\) is the 'insurance instrument' used, or else the element such that for any \(x \in E\) the minimum number of shares \(\alpha \in \mathbb{R}\) are determined such that \(x + \alpha e \in A\).
Through this work we emphasize on:

(a) **the geometric properties of the 'insurance instrument' asset.** In most of the results, $E$ is reflexive, the cone $E_+$ has norm-interior points and $e \in \text{int}E_+$. This case has been studied for non-reflexive spaces in the case of $L_\infty$ (see in Delbaen (2002)).

(b) **the properties of the partial ordering $\geq$ induced by the cone $E_+$.** By the previous assumptions, we may move beyond the case of Banach lattices which are well-known cases of partially ordered spaces in which continuity of risk measures is assured (see Biagini and Fritelli (2009)). We prove continuity for partially ordered spaces whose ordering cones are such that the ordering relation is a non-lattice one. Such a cone is a Bishop-Phelps cone in an infinite-dimensional normed space $E$

$$K(f, a) = \{x \in E \mid f(x) \geq a\|x\|\}, a \in (0, 1), f \in E^*, \|f\| = 1. \quad (1)$$
(c) we prove dual representation results where the representation variable is the spot price functional $\pi$ faced by the investors in the market. This kind of representation for convex risk measures extends the relevant Th.2 of Jaschke and Küchler (2001). Also, indicated cases of coherent risk measures are such the set of representing functionals of a coherent risk measure is bounded such as in the case where the state space is finite.

(d) the fact that for these cases, we prove the Lipschitz continuity of the relevant convex risk measures. This fact implies all the other forms of continuity (e.g. Fatou continuity) mentioned in literature about risk measures.
Coherent and Convex risk measures

Suppose that $E$ is a partially ordered (reflexive) Banach space whose ordering cone is $E_+$. We suppose that the norm-interior of $E_+$ is non-empty and consider $e$ to be such a point. Then a function $\rho : E \to \mathbb{R}$ is called \textit{e-convex} risk measure if it satisfies the following properties:

(i) $\rho(x + \alpha e) = \rho(x) - \alpha$ for all $x \in E$, $\alpha \in \mathbb{R}$ (\textit{e-Translation Invariance})

(ii) $\rho(\lambda x + (1 - \lambda)y) \leq \lambda \rho(x) + (1 - \lambda)\rho(y)$ for all $x, y \in E$, $\lambda \in [0, 1]$ (Convexity)

(iii) $\rho(x) \geq \rho(y)$ if $y \geq x$ with respect to the partial ordering whose positive cone is $E_+$ (\textit{E+-Monotonicity})

If $\rho$ also satisfies the Positive Homogeneity Property, where $\rho(\lambda x) = \lambda \rho(x)$ for all $x \in E$ and $\lambda \in \mathbb{R}_+$, then $\rho$ is called \textit{e-coherent}. 


Dual representation results

**Theorem 1.** If $\rho : E \to \mathbb{R}$ is an $e$-coherent risk measure whose acceptance set $A_\rho$ is $\sigma(E, E^*)$-closed, then

$$
\rho(x) = \sup \{ \pi(-x) \mid \pi \in B \},
$$

for any $x \in E$, where $B = \{ y \in A_\rho^0 \mid \hat{e}(y) = 1 \} = B_e \cap A_\rho^0$.

**Theorem 2.** If $\rho : E \to \mathbb{R}$ is an $e$-convex risk measure with $\sigma(E, E^*)$-closed acceptance set, then

$$
\rho(x) = \sup \{ \pi(-x) - a(\pi) \mid \pi \in B_e \}
$$

(2)

for any $x \in E$, where $B_e = \{ y \in E_+^0 \mid \hat{e}(y) = 1 \}$ and $a : B_e \to \mathbb{R}$ is a 'penalty function' associated with $\rho$, with $a(\pi) \in (-\infty, \infty]$ for any $\pi \in B_e$. The $e$-convexity of every function $\rho$ defined by 2 is obvious.
Remarks on the Duality results

- The theorem mainly used in the proofs is the Strong Separation Theorem for Convex Sets in locally convex spaces.

- The 'penalty function' $\alpha(\pi) = \sup \{ \pi(-x) | x \in A_\rho \}$ where $A_\rho = \{ x \in E | \rho(x) \leq 0 \}$, is defined in a similar way with Th.5 in Föllmer and Schied (2002) for Euclidean spaces of positions.
• If $e = 1$ and $E = L^p(\Omega, \mathcal{F}, P), 1 < p < \infty$, then by considering the set of probability measures

$$Q_{B_1} = \{ Q \in ca(\Omega) | Q << P, \frac{dQ}{dP} = \pi, \pi \in B_1 \},$$

the dual representation of every convex risk measure of the form described in Theorem 2 is

$$\rho(x) = \sup \{ \mathbb{E}_Q(-x) - a(Q) | Q \in Q_{B_1} \}$$

such as in the Th. 5 of Föllmer and Schied (2002).

• In the previous form, since the set of Radon-Nikodym derivatives $\frac{dQ}{dP}$ is bounded, the interiority assumption for 1 provides a case of finiteness and dual representation of convex risk measures on these spaces (see Th.2.11 in Kaina and Rüschendorf (2009))
Continuity results

**Proposition 1.** If $E$ is a reflexive Banach lattice and $\rho : E \to \mathbb{R}$ is an $e$-coherent risk measure with $\sigma(E, E^*)$-closed acceptance set $A_\rho$, then $\rho$ is a Lipschitz function.

**Proposition 2.** If $E$ is reflexive and $\rho : E \to \mathbb{R}$ is an $e$-convex ($e$-coherent) risk measure with $\sigma(E, E^*)$-closed acceptance set $A_\rho$, then $\rho$ is Lipschitz.

**Proposition 3.** If $E$ is a reflexive $L^p$-space, $\rho : E \to \mathbb{R}$ is either an $e$-coherent or an $e$-convex risk measure with $\sigma(E, E^*)$-closed acceptance set $A_\rho$, then $\rho$ is continuous from above, continuous from below, Fatou continuous and Lebesgue continuous.
Definitions of Continuity

A risk functional $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ is called:

- **continuous from above** if for any $(x_n) \subseteq L^p, x_n \downarrow x, P$ a.s. with $x \in L^p$, this implies $\lim_n \rho(x_n) = \rho(x)$.

- **continuous from below** if for any $(x_n) \subseteq L^p, |x_n| \leq y, x, y \in L^p, x_n \uparrow x, P$ a.s. with $x \in L^p$, this implies $\lim_n \rho(x_n) = \rho(x)$.

- **Fatou continuous** if for any $(x_n) \subseteq L^p, |x_n| \leq y, x, y \in L^p, x_n \to x, P$ a.s., this implies $\liminf_n \rho(x_n) \geq \rho(x)$.

- **Lebesgue continuous** if for any $(x_n) \subseteq L^p, |x_n| \leq y, x, y \in L^p, x_n \to x, P$ a.s., this implies $\lim_n \rho(x_n) = \rho(x)$.

About these definitions, see Def.3.1 in Kaina and Rüschendorf (2009).
The Implications of Reflexivity

Suppose that \( E \) is a partially ordered normed linear space. By \( C^{00} \) we denote the subset \( (C^0)^0 \) of \( E^{**} \). If \( C \) is a closed wedge of a reflexive space \( E \), then \( C^{00} = C \). Also, \( f \in E^* \) is a uniformly monotonic functional of \( C \) if and only if \( f \in intC^0 \). If \( E = K - K \) then the wedge \( K \) is called generating. If \( K \) is generating, then \( K^0 \) is a cone of \( E^* \). If \( E \) is reflexive and \( e \) is an interior point of \( E \), then \( E_+ \) is generating, hence \( E_+^0 \) is a cone. According to Th. 4 in Polyak (2008), if \( e \) is an interior point of \( E_+ \) and \( h \in E_+ \) as a linear functional of \( E^* \) being a strictly positive linear functional of \( E_+^0 \), then the base

\[
B_h = \{ \pi \in E_+^0 \mid \hat{h}(\pi) = 1 \}
\]

defined by it is also bounded, hence weakly compact and \( B_e \) is so.
Other results

The norm \( \| . \| \) of \( E \) is called order-unit norm with respect to \( e \) if the following equation holds for any \( x \in E \)

\[
\| x \| = \inf \{ \lambda > 0 | x \in [-\lambda e, \lambda e] \}.
\]

(i) If \( \rho : E \to \mathbb{R} \) is an \( e \)-coherent risk measure and \( \| . \| \) is an order-unit norm with respect to \( e \), \( E_+ \) is a \( \sigma(E, E^*) \)-closed cone and the acceptance set \( A_\rho \) is a cone, then \( \rho \) is Lipschitz. (In this case \( E \) is non-reflexive)

(ii) If \( C \) is a wedge of a normed linear space \( E \) which contains \( E_+ \) and \( e \in \text{int}E_+ \), then \( \rho_C : E \to \mathbb{R} \) is an \( e \)-coherent risk measure which does not take the value \( +\infty \), where

\[
\rho_C(x) = \inf \{ \alpha \in \mathbb{R} | x + \alpha e \in C \}.
\]

Further, if \( C \) is also a cone, then \( \rho_C \) does not take the value \( -\infty \), too. (This result is contained in Jaschke -Küchler (2001) and it we give a detailed proof of it).
Partially ordered linear spaces

Let \( E \) be a (normed) linear space. A set \( C \subseteq E \) satisfying \( C + C \subseteq C \) and \( \lambda C \subseteq C \) for any \( \lambda \in \mathbb{R}_+ \) is called wedge. A wedge for which \( C \cap (-C) = \{0\} \) is called cone. A pair \((E, \geq)\) where \( E \) is a linear space and \( \geq \) is a binary relation on \( E \) satisfying the following properties:

(i) \( x \geq x \) for any \( x \in E \) (reflexive)

(ii) If \( x \geq y \) and \( y \geq z \) then \( x \geq z \), where \( x, y, z \in E \) (transitive)

(iii) If \( x \geq y \) then \( \lambda x \geq \lambda y \) for any \( \lambda \in \mathbb{R}_+ \) and \( x + z \geq y + z \) for any \( z \in E \), where \( x, y \in E \) (compatible with the linear structure of \( E \))

is called partially ordered linear space. The binary relation \( \geq \) in this case is a partial ordering on \( E \).
The set $P = \{x \in E | x \geq 0\}$ is called (positive) wedge of the partial ordering $\geq$ of $E$. Given a wedge $C$ in $E$, the binary relation $\geq_C$ defined as follows:

$$x \geq_C y \iff x - y \in C,$$

is a partial ordering on $E$, called partial ordering induced by $C$ on $E$. 
If the partial ordering $\geq$ of the space $E$ is antisymmetric, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in E$, then $P$ is a cone. $E'$ denotes the linear space of all linear functionals of $E$, while $E^*$ is the norm dual of $E^*$, in case where $E$ is a normed linear space. Suppose that $C$ is a wedge of $E$. A functional $f \in E'$ is called positive functional of $C$ if $f(x) \geq 0$ for any $x \in C$. $f \in E'$ is a strictly positive functional of $C$ if $f(x) > 0$ for any $x \in C \setminus C \cap (-C)$. Some $f \in E'$ where $E$ is a normed linear space, is called uniformly monotonic functional of $C$ if there is some real number $a > 0$ such that $f(x) \geq a\|x\|$ for any $x \in C$. 

In case where a uniformly monotonic functional of $C$ exists, $C$ is a cone. $C^0 = \{ f \in E^* | f(x) \geq 0 \text{ for any } x \in C \}$ is the dual wedge of $C$ in $E^*$. If $C$ is a wedge of $E^*$, then the set $C_0 = \{ x \in E | \hat{x}(f) \geq 0 \text{ for any } f \in C \}$ is the dual wedge of $C$ in $E$, where $\hat{\cdot} : E \to E^{**}$ denotes the natural embedding map from $E$ into the second dual space $E^{**}$ of $E$. Note that if for two wedges $K, C$ of $E$ $K \subseteq C$ holds, then $C^0 \subseteq K^0$. If $C$ is a cone, then a set $B \subseteq C$ is called base of $C$ if for any $x \in C \setminus \{0\}$ there exists a unique $\lambda_x > 0$ such that $\lambda_x x \in B$. The set $B_f = \{ x \in C | f(x) = 1 \}$ where $f$ is a strictly positive functional of $C$ is the base of $C$ defined by $f$. $B_f$ is bounded if and only if $f$ is uniformly monotonic. If $B$ is a bounded base of $C$ such that $0 \notin \overline{B}$ then $C$ is called well-based. If $C$ is well-based, then a bounded base of $C$ defined by a $g \in E^*$ exists. The Theorem 4 by Polyrakis (2008) we use is the following: Suppose that $\langle X, Y \rangle$ is a dual system. If $X$ is a normed space, $P$ a $\sigma(X, Y)$-closed cone of $X$ so that the positive part $U_X^+ = U_X \cap P$ of the closed unit ball $U_X$ of $X$ is $\sigma(X, Y)$-compact, we have: either every base for $P$ defined by a vector $y \in Y$ is bounded or every such base for $P$ is unbounded.
A lattice is a partially ordered linear space $E$ such that for any two elements $x, y \in E$ the supremum and the infimum of them with respect to the partial ordering of it exists. These elements are denoted by $x \vee y, x \wedge y$ respectively. If $E$ is a linear lattice then for any $x \in E$ we may define the positive part $x^+ = x \vee 0$ of $x$, the negative part $x^- = (-x) \vee 0$ of $x$ and the absolute value $|x| = x \vee (-x)$ of $x$. What holds in this case is $x = x^+ - x^-, |x| = x^+ + x^-$. A linear lattice $E$ being also a normed space is called a normed lattice if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ and if this normed space is a Banach space then it is called a Banach lattice.
References


Thank you!