Minimising of Capital Injections by Reinsurance and Investments

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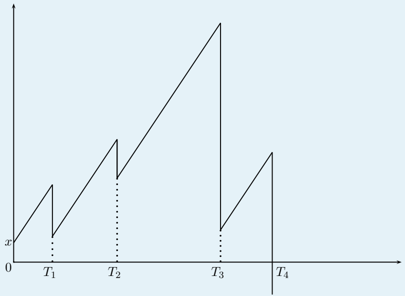
The Classical Model

The Classical Model is a surplus process $X = \{X_t\}$ of the form

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i$$



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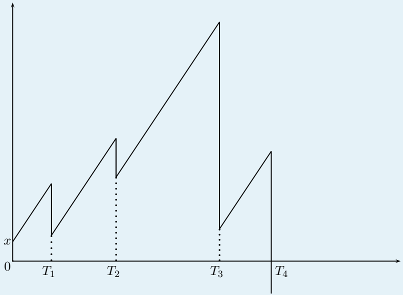

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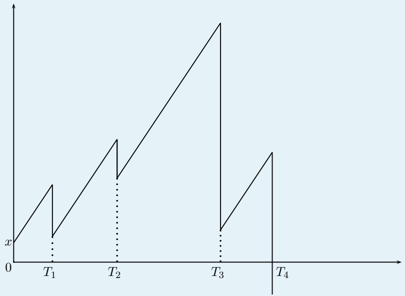

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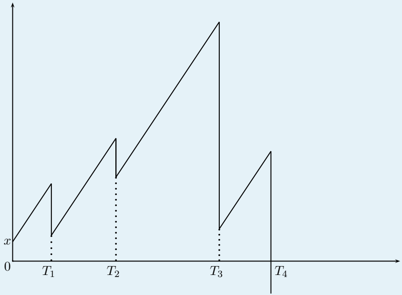

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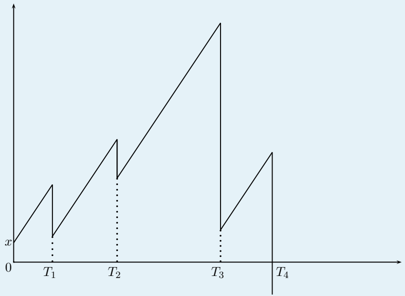

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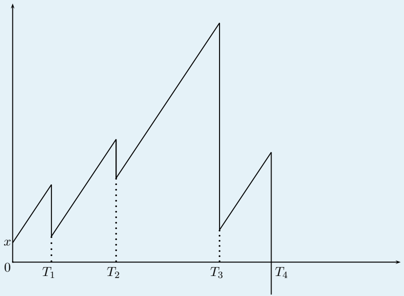

- x: initial capital
- c: premium rate
- Z_i: iid claim sizes
- N_t: Poisson process with intensity λ, independent of Z_i
- T_i: claim arrival times

Reinsurance

Reinsurer $\cong$ is an insurer taking over the risk of another under contract.

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Problem of the first insurer:

How much reinsurance should I buy?

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- the self-insurance function $r(z, b)$. We assume that r is continuous and increasing in both variables;

Reinsurance

Problem of the first insurer:

How much reinsurance should I buy?

The decision depends on

- the retention level $b \in [0, \tilde{b}]$;
- the self-insurance function $r(z, b)$. We assume that r is continuous and increasing in both variables;
- the premium rate $c(b)$. $c(b)$ denotes the premium remaining to the first insurer, if the retention level b was chosen.

Investments

We allow the insurer to invest into a risky asset, modeled as a Black-Scholes model

$$\mathrm{d}Q_t = mQ_t\,\mathrm{d}t + \sigma Q_t\,\mathrm{d}W_t \quad \Leftrightarrow \quad Q_t = \exp\{(m - \frac{\sigma^2}{2})t + \sigma W_t\}\ ,$$

where m and $\sigma > 0$ are constants and $W = \{W_t\}$ a standard Brownian motion.

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We are not interested in asset price, but in asset return!

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where m and $\sigma > 0$ are constants and $W = \{W_t\}$ a standard Brownian motion.

The return of such a process is the stochastic process $\{Q'_t\}$ given by the stochastic differential equation

$$\mathrm{d}Q'_t = \frac{\mathrm{d}Q_t}{Q_t} = m \,\mathrm{d}t + \sigma \,\mathrm{d}W_t\ .$$

Surplus Process with Investments, Reinsurance and Capital Injections

$$X_t^{A,B} = x + \int_0^t c(b_s)\, \mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-}) + m \int_0^t a_s\, \mathrm{d}s + \sigma \int_0^t a_s\, \mathrm{d}W_s$$

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- $m\int_0^t a_s\,\mathrm{d}s + \sigma\int_0^t a_s\,\mathrm{d}W_s$: asset return.

Surplus Process with Investments, Reinsurance and Capital Injections

$$X_t^{A,B,Y} = x + \int_0^t c(b_s)\ \mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-}) + m \int_0^t a_s\ \mathrm{d}s + \sigma \int_0^t a_s\ \mathrm{d}W_s + Y_t^{A,B}$$

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Return and Value Functions

Assumptions:

- the processes $\{\sum_{i=1}^{N_t} Z_i\}$ and $\{W_t\}$ are independent;
- the filtration $\{\mathcal{F}_t\}$ is generated by the pair $(\sum_{i=1}^{N_t} Z_i, W_t)$;
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$$\underbrace{V(x)}_{\text{value function}} = \inf_{(A,B)} \underbrace{V^{A,B}(x)}_{\text{return function}} = \inf_{(A,B)} \mathbb{E}_x\Big[\int_0^\infty e^{-\delta t}\, \mathrm{d}Y_t^{A,B}\Big].$$

Constant Strategies

$$X_t^{a,b} = x + \big(c(b) + am\big)t - \sum_{i=1}^{N_t} r(Z_i, b) + a\sigma W_t$$

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Let Z_i be exponentially distributed, $b = \tilde{b}$ and $\mathbb{E}[Z_i] = \mu$, then it holds with $R < 0$ given by $\delta + \lambda + cR = \int_0^\infty e^{R \cdot z} \, \mathrm{d}G(z)$:

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The derivation of the corresponding return function becomes much more complicated!

Lemma 1

The value function $V(x)$ has the following properties

1. $V(x)$ is monotone decreasing $\lim_{x\to\infty} V(x) = 0$;
2. $V(x)$ is Lipschitz continuous on $[0, \infty)$ with $|V(x) - V(y)| \leq |x - y|$;

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1. follows directly from $\lim_{x\to\infty} V^{0,\tilde{b}}(x) = 0$.
2. Let $z > x$ and (A, B) be a strategy with $V^{A,B}(z) \leq V(z) + \epsilon$, $\epsilon > 0$. Construct a strategy (A', B') for the initial capital x as follows: inject the capital $z - x$ and then follow the strategy (A, B).
   Then:

$$V(x) - V(z) \leq V^{A',B'}(x) - V^{A,B}(z) + \epsilon = z - x + \epsilon \,.$$

The Value Function is Convex

Remark 1

If $c(b)$ is concave in b and $r(z, b) = zb$, $b \in [0, 1]$, then $V(x)$ is convex.

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Proof:

Let $x, z \geq 0$, $\alpha \in (0,1)$ and $y = \alpha x + (1-\alpha)z$. Let further (A^x, B^x) be the optimal strategy for initial capital x, (A^z, B^z) for z and (A^y, B^y) for y. Then it holds

$$\begin{aligned} & X_t^{A^y,B^y,Y} - Y_t^{A^y,B^y} + \alpha Y_t^{A^x,B^x} + (1-\alpha) Y_t^{A^z,B^z} \\ \geq \quad & \alpha X_t^{A^x,B^x,Y} + (1-\alpha) X_t^{A^z,B^z,Y} \geq 0 \,. \end{aligned}$$

Because $\int_0^\infty e^{-\delta t}\, \mathrm{d}Y_t = \delta \int_0^\infty e^{-\delta t} Y_t \, \mathrm{d}t$ we obtain

$$V(\alpha x + (1-\alpha)z) = V(y) \leq \alpha V(x) + (1-\alpha) V(z) \,.$$

Hamilton–Jacobi–Bellman Equation

The HJB equation of the considered problem is

$$0 = \inf_{\substack{a \in \mathbb{R} \\ b \in [0,\tilde{b}]}} \frac{\sigma^2 a^2}{2} V''(x) + \lambda \int_0^\infty V(x - r(z,b)) \, \mathrm{d}G(z) + (c(b) + am)V'(x) - (\delta + \lambda)V(x) \,. \tag{1}$$

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Assume $V(x)$ is twice continuously differentiable. Minimising with respect to a yields

$$0 = \inf_{b \in [0,\tilde{b}]} \left\{ \lambda \int_0^\infty V(x - r(z,b)) \, \mathrm{d}G(z) + c(b)V'(x) \right\} - \frac{m^2 V'(x)^2}{2\sigma^2 V''(x)} - (\delta + \lambda)V(x) \,.$$

In particular: If $V(x)$ is twice continuously differentiable, then the optimal investment strategy is given by

$$a^*(x) = -\frac{mV'(x)}{\sigma^2 V''(x)} .$$

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$$\Downarrow$$

A **viscosity solution** to (1) is a continuous function $u : [0, \infty) \to \mathbb{R}_+$ if it is both a viscosity subsolution and a viscosity supersolution at any $x \in (0, \infty)$.

Viscosity Subsolution

A continuous function $\underline{u} : [0, \infty) \to \mathbb{R}_+$ is a **viscosity subsolution** to (1) at $x \in (0, \infty)$ if any twice continuously differentiable function $\psi : (0, \infty) \to \mathbb{R}$ with $\psi(x) = \underline{u}(x)$ such that $\underline{u} - \psi \leq 0$ satisfy

$$0 \leq -\frac{m^2 \psi'(x)^2}{2\sigma^2 \psi''(x)} + \inf_{b \in [0, \tilde{b}]} \left\{ \lambda \int_0^\infty \underline{u}(x - r(z,b))\, \mathrm{d}G(z) + c(b)\psi'(x) \right\} - (\delta + \lambda)\underline{u}(x) \; .$$

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Viscosity Supersolution

A continuous function $\bar{u} : [0, \infty) \to \mathbb{R}_+$ is a **viscosity supersolution** to (1) at $x \in (0, \infty)$ if any twice continuously differentiable function $\phi : (0, \infty) \to \mathbb{R}$ with $\phi(x) = \bar{u}(x)$ such that $\bar{u} - \phi \geq 0$ satisfies

$$0 \geq -\frac{m^2 \phi'(x)^2}{2\sigma^2 \phi''(x)} + \inf_{b \in [0, \tilde{b}]} \left\{ \lambda \int_0^\infty \bar{u}(x - r(z, b)) \, \mathrm{d}G(z) + c(b)\phi'(x) \right\} - (\delta + \lambda)\bar{u}(x) \; .$$

Main Result

Theorem 1

$V(x)$ is a viscosity solution to (1).

Theorem 2: Comparison Principle

let $v(x)$ be a super- and $u(x)$ a subsolution to (1), fulfilling conditions 1. and 2. of Lemma 1. If it holds $u(0) \leq v(0)$, then $u(x) \leq v(x)$ on $[0, \infty)$.

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Conclusion: The value function $V(x)$ is the unique viscosity solution to the HJB equation (1).

The Hamilton–Jacobi–Bellman Equation

We consider the case $r(z,b) = zb$, $b \in [0,1]$, i.e. the proportional reinsurance

The HJB equation is

$$\inf_{\substack{a\in\mathbb{R} \\ b\in[0,\tilde{b}]}} \frac{\sigma^2 a^2}{2} V''(x) + \lambda \int_0^\infty V(x - r(z,b))\,\mathrm{d}G(z) + (c(b) + am)V'(x) - \lambda V(x) = 0\,. \tag{2}$$

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In this case we can show that the value function is the unique twice continuously differentiable solution to the HJB equation (2).

From the HJB equation we obtain for $c(b) = -\lambda\mu(\theta - \eta) + \lambda\mu(1 + \theta)b$

$V''(x) > 0$: $V''(x) \geq 0$ follows from the convexity of the value function; $V''(x) \neq 0$ follows from

$$\frac{\sigma^2 a^2}{2} V''(x) + \big(c(0) + am\big)V'(x) < 0 \quad \text{for} \quad a = \frac{1 - c(0)}{m} .$$

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The optimal strategy at $x = 0$ is given by

$$(a^*, b^*) = \begin{cases} \left(\frac{2\lambda\mu(\theta-\eta)}{m}, 0\right) & : \ V'(0) \in \left(-\frac{1}{1+\theta}, 0\right) , \\ (a^*, 1) & : \ V'(0) \in \left(-1, -\frac{1}{1+\theta}\right) , \\ \left(\frac{2\lambda\mu(\theta-\eta)}{m}, b\right) & : \ V'(0) = -\frac{1}{1+\theta} \quad , b \in [0, 1] . \end{cases}$$

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If $V'(0) \in (-1, -\frac{1}{1+\theta})$, then it holds $b^*(x) = 1$ for $x \in [0, \epsilon)$, $\epsilon > 0$.

Verification Theorem and The Optimal Strategy at $x = 0$

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Let $f(x)$ be a decreasing and twice continuously differentiable solution to (2) with $\lim_{x\to\infty} f(x) = 0$. Then it holds $f(x) = V(x)$ and the optimal strategy is of the feedback form $(A^*(X_t), B^*(X_t))$.

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From Lemma 2 it follows that the optimal reinsurance strategy at $x = 0$ is given by $b^* = 1$.

Existence of a Two Times Differentiable Solution

Theorem 4

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The exponential distribution $G(x) = 1 - e^{-x/\mu}$ and the Pareto distribution $G(x) = 1 - \frac{\mu^2}{(\mu+x)^2}$ with $c(b) = \lambda\mu(\eta - \theta) + \lambda\mu(1+\theta)b$ satisfy the conditions of Theorems 4.

The surplus process under investments, reinsurance and with capital injections fulfils

$$X_t^{A,B,Y} = x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-}) + m\int_0^t a_s\,\mathrm{d}s + \sigma\int_0^t a_s\,\mathrm{d}W_s + Y_t^{A,B}$$

where

- $A = \{a_t\}$ and $B = \{b_t\}$, $a_t \in \mathbb{R}$, $b_t \in [0, \tilde{b}]$ admissible investment and reinsurance strategies;
- $\int_0^t c(b_s)\,\mathrm{d}s$ premia until t;
- $\sum_{i=1}^{N_t} r(Z_i, b_{T_i-})$ claims until t;
- $m\int_0^t a_s\,\mathrm{d}s + \sigma\int_0^t a_s\,\mathrm{d}W_s$: asset return.

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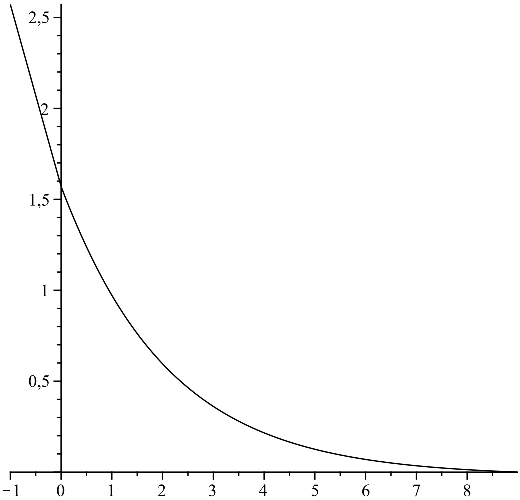
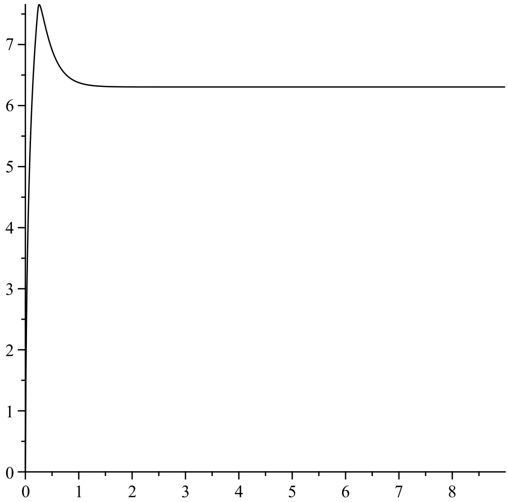
For the parameters $\sigma^2 = 0.01$, $m = 0.03$, $\delta = 0.04$, $\mu = \lambda = 1$, $\eta = 0.3$ and $\theta = 0.5$ we can calculate the value function and the optimal strategy pair (A^*, B^*) numerically.

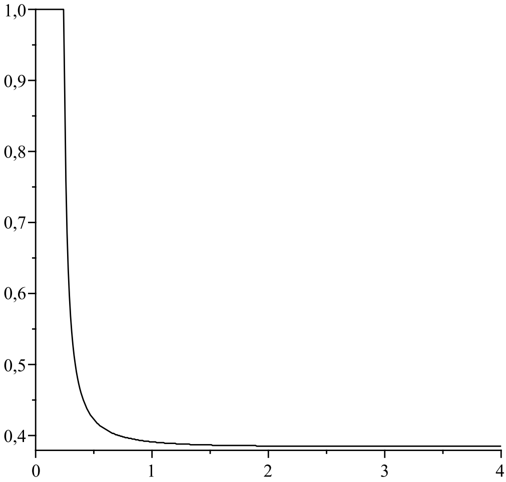
The Value Function $V(x)$ for $G(x) = 1 - e^{-\frac{x}{\mu}}$



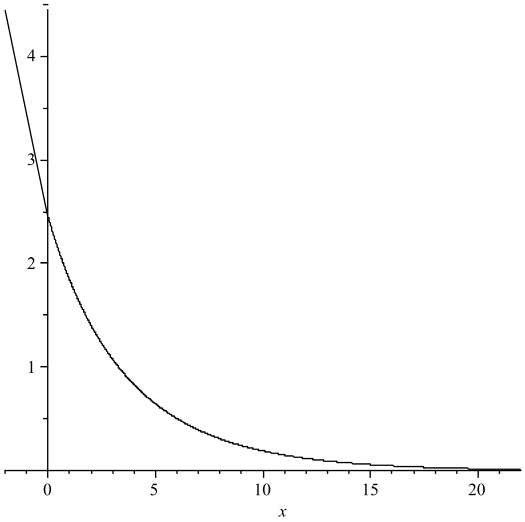
The Optimal Investment Strategy A^*



The Optimal Reinsurance Strategy B^*



The Value Function $V(x)$ for $G(x) = 1 - \frac{\mu^2}{(\mu+x)^2}$



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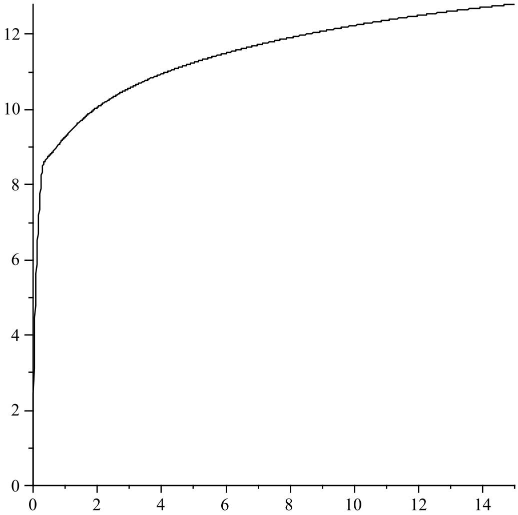
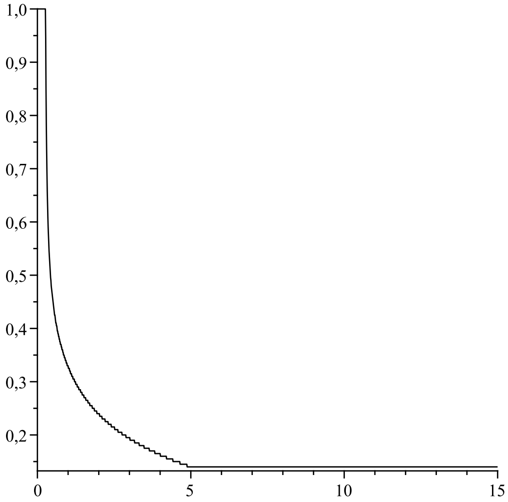


The Optimal Reinsurance Strategy B^*



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Thank You for Your Attention