Multivariate Extremes, Geometry & the Financial Crisis*:

*Warnings, guilt and lessons hopefully learned

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This talk is very much based on the following RiskLab publications:

Catherine Donnelly and Paul Embrechts:
The devil is in the tails: actuarial mathematics and the subprime crisis
Astin Bulletin 40(1), 1 - 33, 2010

Guus Balkema, Paul Embrechts, Natalia Nolde:
Meta densities and the shape of their sample clouds
J. Multivariate Analysis 101, 1738 -1754, 2010

Warnings (two examples)

Embrechts, P. et al. (2001): An academic response to Basel II. Financial Markets Group, London School of Economics. (Mailed to the Basel Committee)

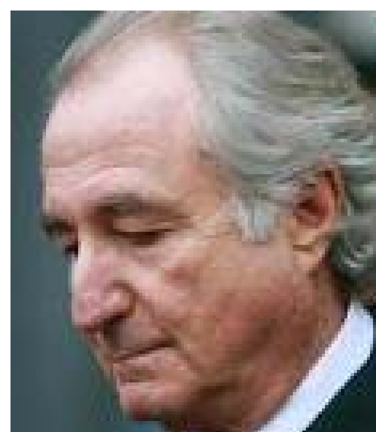
(Critical on VaR, procyclicality, systemic risk, RAs, ...)

Markopolos, H. (2005): The world’s largest hedge fund is a fraud. (Mailed to the SEC)

Charles Ponzi
1910

(Madoff runs a Ponzi scheme)

Bernard Madoff



olos

An interesting story!

"It's like the universe wanted to dish out a reward when Madoff was finally sent to prison. That reward went to Ralph Amendolaro of Queens's, New York.

On March 15, Amendolaro won $1,500 playing the last three numbers of the gargantuan crook's prison number -- 61727-054A. As strings of numbers go, that one's creepy." (local newspaper)

Guilt (first mathematics and then the real ones)

Recipe for Disaster: The Formula That Killed Wall Street

By Felix Salmon 23 February, 2009

Wired Magazine

$$\Pr[T_A<1, T_B<1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1), \gamma)$$

Error,)

I (we) did notice and warned very early on!

28 March 1999 (!!!)
Columbia-JAFEE Conference on the Mathematics of Finance, Columbia University, New York.

10:00-10:45 P. Embrechts (ETH, Zurich):

Insurance Analytics:
Actuarial Tools in Financial Risk Management

Based on P. Embrechts, A.J. McNeil, D. Straumann (1999)
Correlation and Dependence in Risk Management:
Properties and Pitfalls. Preprint RiskLab/ETH Zürich (!!!)

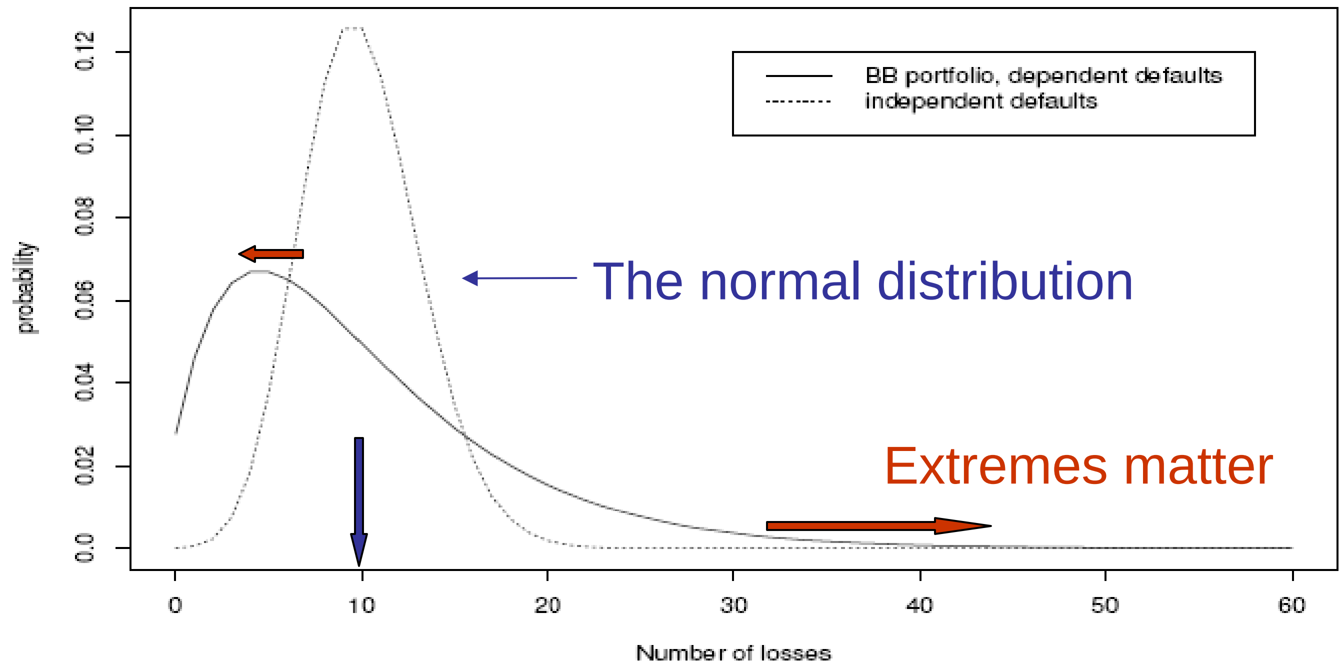
Coffee break: discussion with David Li

Early work goes back to 1960's...

Masaaki Sibuya

micro-

Impact of dependence on loss distribution



Distribution of number of defaults for homogeneous portfolio of 1000 BB loans with default probability $\approx 1\%$; Bernoulli mixture model with default correlation $\approx 0.22\%$ is compared with independent default model.

Correlation matters

Guilt: the real culprits!

- LCFIs and “securitization”
- REPO 105
- Regulatory arbitrage
- And of course: … many more!

A Mathematical Theorem
(Multivariate Extremes and Geometry)

Definition (Meta distribution)

- Random vector $\mathbf{Z}$ in $\mathbb{R}^d$ with df F and continuous marginals F_i, $i = 1, \dots, d$
- $G_1, \dots, G_d$: continuous df's on $\mathbb{R}$, strictly increasing on $\mathrm{I_i} = \{0 < \mathrm{G_i} < 1\}$
- Define transformation:

$$K(x_1, \dots, x_d) = \big(K_1(x_1), \dots, K_d(x_d)\big), \quad K_i(s) = F_i^{-1}\big(G_i(s)\big), \quad i = 1, \dots, d$$

- The df $G = F \circ K$ is the meta distribution (with marginals G_i) based on original df F
- $\mathbf{X}$ is said to be a meta vector for $\mathbf{Z}$ (with marginals G_i) if $\mathbf{Z} \stackrel{\mathrm{d}}{=} K(\mathbf{X})$
- The coordinatewise map $K = K_1 \otimes \cdots \otimes K_d$ which maps $\mathbf{x} = (x_1, \dots, x_d) \in \mathrm{I} = \mathrm{I_1} \times \cdots \times \mathrm{I_d}$ into the vector $\mathbf{z} = (K_1(x_1), \dots, K_d(x_d))$ is called the meta transformation

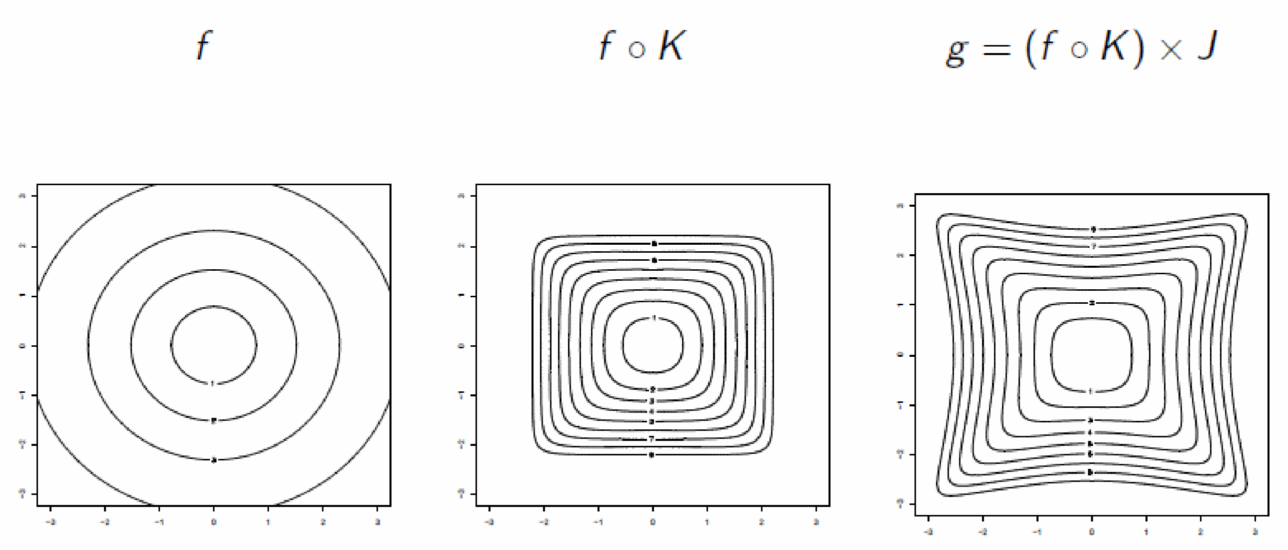
Proposition

If

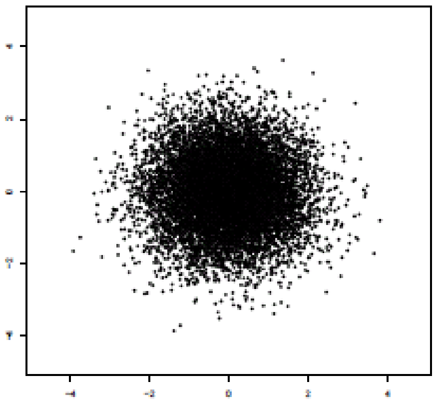
- Original vector $\mathbf{Z}$ has a density, f
- Marginals of meta distribution have densities, g_i

then the meta distribution has a density, g, and g is of the form

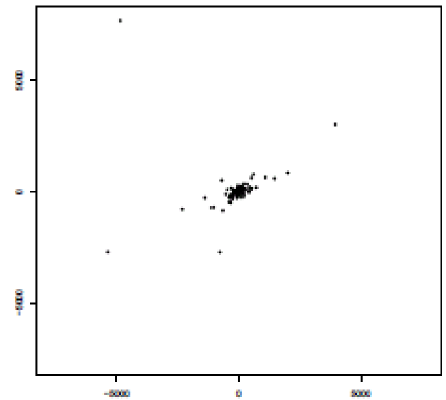
$$g(\mathbf{x}) = f(K(\mathbf{x})) \prod_{i=1}^{d} \frac{g_i(x_i)}{f_i(z_i)} \qquad z_i = K_i(x_i),\ x_i \in \mathrm{I_i} = \{0 < \mathrm{G_i} < 1\}$$



Examples of sample clouds

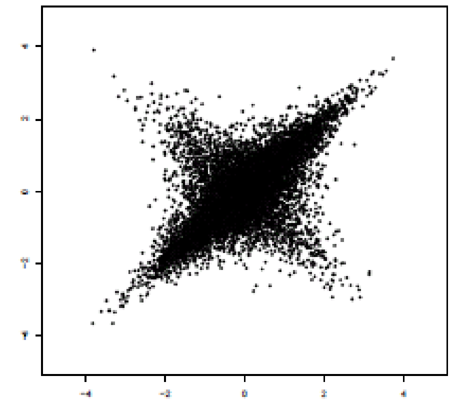


standard normal



elliptic Cauchy with dispersion matrix

$$\Sigma = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$



meta-Cauchy with normal marginals

Definition (Standard set-up)

d → • f: a continuous density on $\mathbb{R}^d$, positive outside a bounded set

• $f(\mathbf{z}) \sim f_0(n_D(\mathbf{z}))$ for $\|\mathbf{z}\| \to \infty$, where

  - f_0: continuous, strictly decreasing
λ → - $f_0 \in RV_{-(\lambda+d)}$
  - D: bounded star-shaped open set containing the origin, with continuous boundary

• meta density g with marginal densities g_d satisfying

  - g_d: continuous, positive, symmetric
  - $g_d \sim e^{-\psi}$, a von Mises function; i.e.

$$\psi'(s) > 0, \quad \psi'(s) \to \infty, \quad (1/\psi')'(s) \to 0 \qquad s \to \infty$$

θ → • Additional condition: $\psi \in RV_\theta, \quad \theta > 0 \qquad (\bigstar)$

• Remarks:

  - $(\bigstar)$ is necessary to have a limit shape
  - $(\bigstar)$ is satisfied for normal, Laplace, Weibull densities and densities of the form $g_d(s) \sim as^b e^{-ps^\theta}$, $s \to \infty$, $a, p, \theta > 0$

The limit function χ

- First assume $f(\mathbf{z}) = f_0(\|\mathbf{z}\|_\infty)$ for a continuous strictly decreasing function $f_0 \in RV_{-(\lambda+d)}$
- Under the standard set-up & ($\bigstar$), it can be shown for $v = \|\mathbf{u}\|_\infty > 0$

$$\chi_s(\mathbf{u}) \to \chi(\mathbf{u}) = |u_1|^\theta + \cdots + |u_d|^\theta + \lambda - (\lambda + d)v^\theta \qquad s \to \infty$$

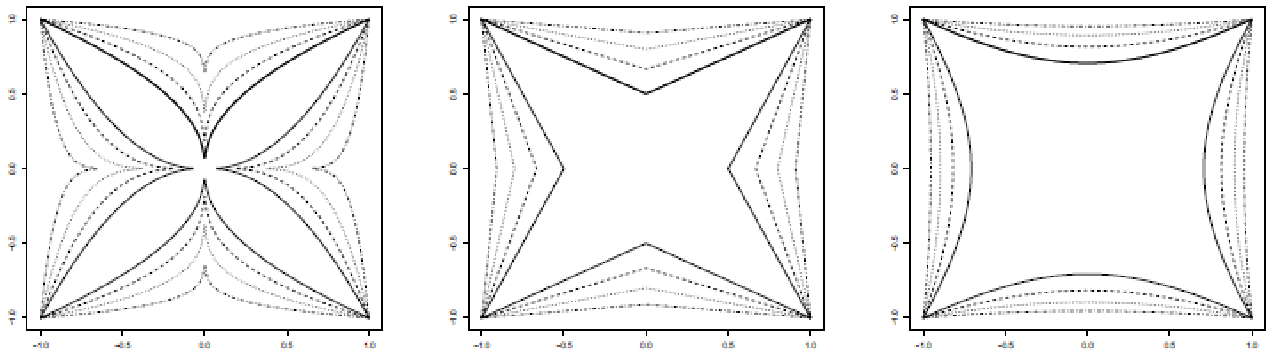
  - Convergence is uniform on $\Pi_r \setminus \epsilon B$ for any $r \geq 1$ and $\epsilon > 0$, where $\Pi_r = \{\mathbf{x} \mid |x_i| \leq x_d \leq r\}$ (upside down pyramid)

- Hence, the limit set is given by

(d, λ, θ) → $$E := E_{\lambda,\theta} := \{\mathbf{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\} \mid |u_1|^\theta + \cdots + |u_d|^\theta + \lambda \geq (\lambda + d)\|\mathbf{u}\|_\infty^\theta\} \qquad (\clubsuit)$$

Examples of limit sets

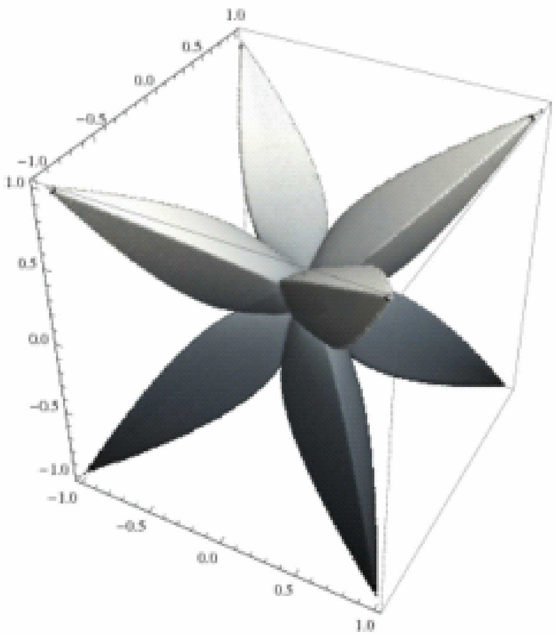
$\theta = 0.1$ $\qquad$ $\theta = 1$ $\qquad$ $\theta = 2$



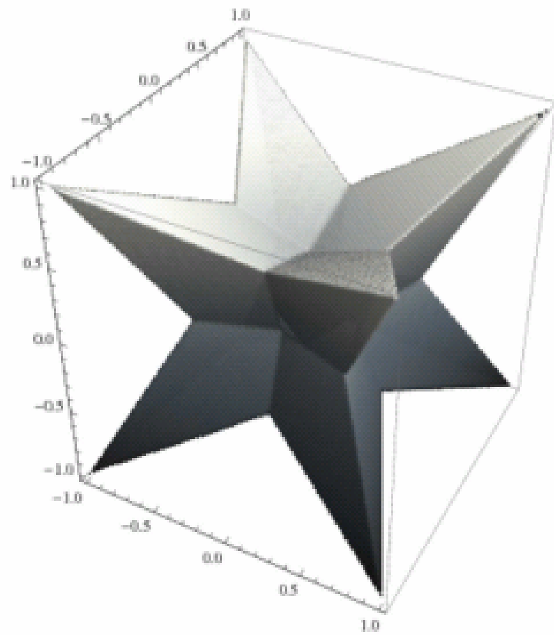
Legend: $\lambda = 1$ (solid), $\lambda = 2$ (dashed), $\lambda = 4$ (dotted), $\lambda = 10$ (dotdash)

Examples of limit sets $E_{\lambda,\theta}$ in $\mathbb{R}^3$ $(\lambda = 1)$

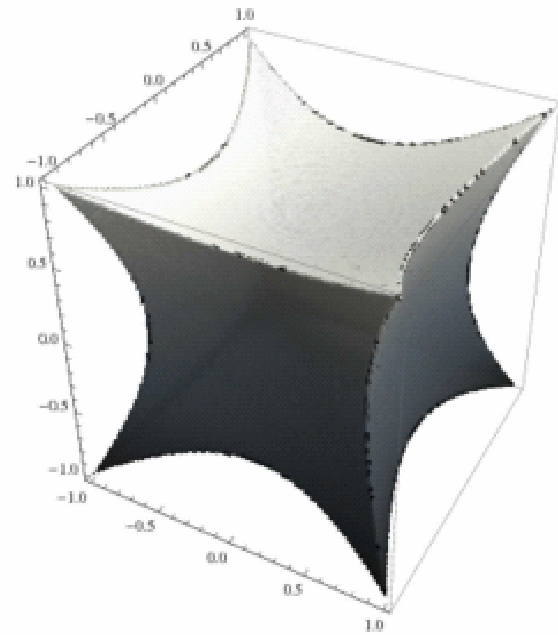
$\theta = 0.5$



$\theta = 1$



$\theta = 2$



Sample clouds: Notation

- $\mathbf{X}_1, \mathbf{X}_2, \ldots$ i.i.d. random vectors on $\mathbb{R}^d$
- $N_n = \{\mathbf{X}_1/a_n, \ldots, \mathbf{X}_n/a_n\}$: n-point sample cloud with scaling constants $a_n > 0$, $a_n \to \infty$ as $n \to \infty$
- $N_n(A) = \sum_{i=1}^n \mathbf{1}_A(\mathbf{X}_i/a_n)$ for any Borel set $A \subset \mathbb{R}^d$
  $\uparrow$
  number of points of N_n contained in set A

Sample clouds: Convergence onto a set

Definition

The sample clouds N_n converge onto a compact set E in $\mathbb{R}^d$ if

- $\mathbb{P}\{N_n(U^c) > 0\} \to 0$ *for open sets U containing E, and*
- $\mathbb{P}\{N_n(\mathbf{p} + \epsilon B) > m\} \to 1,\ m \geq 1,\ \epsilon > 0,\ \mathbf{p} \in E$

  where B denotes the unit Euclidean ball

The set E is called a limit set

We show:

- For sample clouds from the meta density g there is a limit shape
- If
  - $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is a random sample from meta density g
  - Scaling factor r_n is chosen s.t. $ng(r_n\mathbf{1}) \to 1$

  then the scaled sample cloud $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ roughly fills out the limit set E

Main result

Theorem

Let:

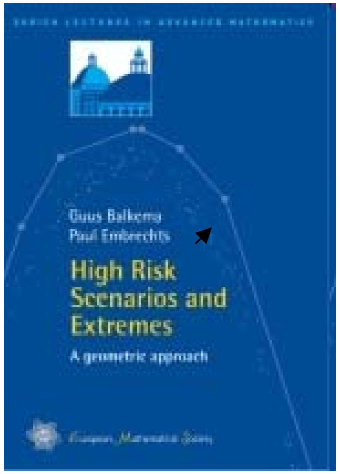
- f & g_d satisfy assumptions of the standard set-up & (★)
- g: meta density with marginals g_d based on original density f
- $r_n > 0$ satisfies $g_d(r_n) \sim 1/n$
- $E = E_{\lambda,\theta}$: closed subset of $C = [-1, 1]^d$ defined in (♣)

Then:

- Level sets $\{g \geq 1/n\}$ scaled by r_n converge to E
- For any sequence of independent observations $\mathbf{X}_n$ from meta density g, the scaled sample clouds $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ converge onto E

Numerous further results:

- Other examples
- More specific convergence properties
- (non-)Robustness/sensitivity results
- An alternative approach to MEVT
- Interplay Geometry – Probability
- Statistical estimation
- …



Thank you!