Reinsurance and ruin problem: asymptotics in the case of heavy-tailed claims

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In a simple insurer–reinsurer model, one splits the claim φ into $G_{\mathrm{I}}(\varphi)$ and $G_{\mathrm{R}}(\varphi)$ where $G_{\mathrm{I}}(x)$ and $G_{\mathrm{R}}(x)$ are the retention and the compensation function respectively. Let η be a generic inter-claim time. We consider a two-dimensional random walk whose increments are i.i.d. copies of $(G_{\mathrm{I}}(\varphi) - \eta, G_{\mathrm{R}}(\varphi) - \eta)$. Various asymptotics for the 'ruin' in infinite time are derived in the case where φ has a heavy-tailed distribution.

Dickson and Waters (1997) optimise reinsurance by minimising the probability of ruin of the insurer in finite and discrete or continuous time, subject to the expected net profit being constant, see also Hald and Schmidli (2004). Schmidli (2004, 2005) studied the asymptotics of ruin probabilities for the insurer risk processes under optimal reinsurance policies, both for large and small claim sizes.

In this talk, we do not study optimality questions and, for a fixed 'proportional' insurer-reinsurer treaty, consider a joint claim surplus process of the insurer and the reinsurer and describe the most likely scenarios for system's 'ruin'. We assume that the claims have a subexponential integrated tail distribution.

Basics.

Suppose that consecutive i.i.d. claims of sizes $\varphi_1, \varphi_2, \ldots$ are arriving according to a Poisson process $N(t)$. We denote the common distribution of the claim sizes by F. Without loss of generality we may assume that the claim arrival intensity $\lambda = 1$, then the inter-claim times $\eta_1, \eta_2, \ldots$ form a sequence of i.i.d. random variables having exponential distribution with parameter 1. We assume that (φ_i) and (η_i) are mutually independent. In the risk theory one considers the *claim surpluss process*

$$S(t) = \sum_{j=1}^{N(t)} \varphi_j - pt,$$

where $p > 0$ is the premium rate. For an initial capital $u > 0$, the *ruin time* is

$$\tau = \tau(u) = \inf\{t > 0 : S(t) > u\}$$

and the *ruin function* is

$$\psi(u) = \mathbb{P}(\tau(u) < \infty).$$

Let $S_n = \sum_{j=1}^n \xi_j$, $n \geq 1$ be a random walk with i.i.d. increments $\xi_j = \varphi_j - p\eta_j$.
Let

$$M = \sup_{t \geq 0} S(t) = \sup_{n \geq 0} S_n.$$

Then the ruin function is equal to

$$\psi(u) = \mathbb{P}(M > u) = \mathbb{P}(L(u) < \infty)$$

where $L(u) = \min\{n : \ S_n > u\}$.

We assume the distribution F of φ to be heavy-tailed,

$\mathbf{E}e^{t\varphi} = \infty$, for all $t > 0$,

and, moreover, to have a subexponential integrated tail distribution,

$$\overline{F}^s(x) = \min\left(1, \int_x^\infty \overline{F}(t)dt\right),$$

for $x > 0$, and $\overline{F}(x) = 1$ for $x \leq 0$.

Assume further that

$$-a := \mathbf{E}\xi < 0.$$

Then (cf. Borovkov, Cohen, Pakes, Veraverbeke, ...)

$$\psi(u) \sim \mathbb{P}\left(\bigcup_{n \geq 1} \{\varphi_n > x + na\}\right) \sim \sum_{n \geq 1} \mathbb{P}(\varphi_n > x+na) \sim \frac{1}{a}\overline{F}^s(u), \quad u \to \infty.$$

This is the *principle of a single big jump*.

A multidimensional extention of the ruin problem has been studied recently, for instance by Avram, Palmowski and Pistorius (2008) (see also references therein), where some formulas and asymptotics have been given in the case of a light-tail distribution of φ. In the two or higher dimensional setting, one may have the following actuarial motivations.

The model.

We assume that, for any single claim φ, the insurer pays $G_{\mathrm{I}}(\varphi)$ and the rest $G_{\mathrm{R}}(\varphi)$ is covered by the reinsurer. Here $G_{\mathrm{I}}(x)$ is a *retention function* and $G_{\mathrm{R}}(x)$ a *compensation function*. We assume that retention function $G_{\mathrm{I}}(x)$ and compensation function $G_{\mathrm{R}}(x)$ are nonnegative and increasing, and clearly $G_{\mathrm{I}}(x) + G_{\mathrm{R}}(x) = x$. The insurer is willing to buy a reinsurance with retention function $G_{\mathrm{I}}(x)$ and to pay premium with intensity $p_{\mathrm{I}} = (1 + \varepsilon_{\mathrm{I}})\mathbb{E}G_{\mathrm{I}}(\varphi)$. Let

$$p_{\mathrm{I}} + p_{\mathrm{R}} = p$$

where $p_{\mathrm{R}} = (1 + \varepsilon_{\mathrm{R}})\mathbb{E}G_{\mathrm{R}}(\varphi)$. Here ε_{I} and ε_{R} are called *relative safety loading* for the insurer and the reinsurer respectively. We assume that safety loadings are strictly positive.

The claim surpluss process of the insurer ($\mathrm{i} = \mathrm{I}$) or the reinsurer ($\mathrm{i} = \mathrm{R}$) is

$$S_{\mathrm{i}}(t) = \sum_{j=1}^{N(t)} G_{\mathrm{i}}(\varphi_j) - p_{\mathrm{i}} t$$

and, for a given initial capital u, the *ruin time* is

$$\tau_{\mathrm{i}}(u) = \inf\{t : \ S_{\mathrm{i}}(t) > u\}.$$

Let $S_{\mathrm{i},n} = \sum_{j=1}^{n} \xi_{\mathrm{i},j}$, $n = 1, 2, \ldots$ be a random walk with i.i.d. increments $\xi_{\mathrm{i},j} = G_{\mathrm{I}}(\varphi_j) - p_{\mathrm{i}}\eta_j$ with mean $-a_{\mathrm{i}} = \mathbf{E}\xi_{\mathrm{i},j}$. We define

$$M_{\mathrm{i}} = \sup_{t \geq 0} S_{\mathrm{i}}(t) = \sup_{n \geq 0} S_{\mathrm{i},n}.$$

The ruin functions (for $i = I, R$) are

$$\psi_{\mathrm{i}}(u) = \mathbb{P}(\tau_{\mathrm{i}}(u) < \infty) = \mathbb{P}(M_{\mathrm{i}} > u) = \mathbb{P}(L_{\mathrm{i}} < \infty)$$

where

$$L_{\mathrm{i}}(u) = \min\{n : \ S_{\mathrm{i},n} > u\}.$$

We formulate below some results for the infinite time horizon and for the proportional insurance-reinsurance only. Similar results hold for finite time intervals, but under stronger assumptions on the claim size distribution.

Results.

We consider the two-dimensional random walk $(S_{\mathrm{I},n}, S_{\mathrm{R},n})$, $n = 1, 2, \ldots$.

We assume the whole system to be stable, $\varepsilon_{\mathrm{I}} > 0$ and $\varepsilon_{\mathrm{R}} > 0$. The stability condition is equivalent to $\mathbb{E}G_{\mathrm{I}}(\varphi) < p_{\mathrm{I}} < p - \mathbb{E}\varphi + \mathbb{E}G_{\mathrm{I}}(\varphi)$.
We consider the following ruin functions in the infinite time horizon:

- probability that at least one component of the system is ruined

$$\psi_{\mathrm{I}\vee\mathrm{R}}(u,v) \quad = \quad \mathbb{P}(L_{\mathrm{I}}(u) < \infty \quad \text{or} \quad L_{\mathrm{R}}(v) < \infty),$$

- probability that both components of the system are ruined

$$\psi_{\mathrm{I}\wedge\mathrm{R}}(u,v) = \mathbb{P}(L_{\mathrm{I}}(u) < \infty, L_{\mathrm{R}}(v) < \infty),$$

- probability of ruin of the system by the same claim

$$\psi_{\mathrm{I}=\mathrm{R}}(u,v) \quad = \quad \mathbb{P}(L_{\mathrm{I}}(u) = L_{\mathrm{R}}(v) < \infty).$$

We find the asymptotics for $\psi_{\mathrm{I}\wedge\mathrm{R}}(u,v)$ and $\psi_{\mathrm{I}=\mathrm{R}}(u,v)$ in terms of marginal distribution tails. Then the asymptotics for $\psi_{\mathrm{I}\vee\mathrm{R}}(u,v)$ follows.

Recall that any subexponential distribution is *long-tailed*.

A distribution function F with an unbounded (from the right) support is *long-tailed* if

$$\frac{\overline{F}(x+1)}{\overline{F}(x)} \to 1, \quad x \to \infty.$$

A direct consequence of the long-tailedness and of the monotonicity of the distribution function is the following property: if F is long-tailed, then there exists a non-negative monotone increasing function $h(x) \to \infty$ such that $\overline{F}(x+y) \sim \overline{F}(x)$ uniformly in $|y| \leq h(x)$, as $x \to \infty$. Then we say that F is h-*insensitive*.

One can use the concept of h-insensitivity for *any* non-negative function h, which is not necessarily increasing.

Condition I. $u/v \to \mu \in (0,\infty)$ as $u, v \to \infty$.

Condition II.

(IIa). As $x \to \infty$, $G_I(x)/x \to c \in (0,1)$ and

(IIb) F^s is h-insensitive, with

(1) $$h(x) = sup_{0\le y\le x} \left| G_I^{-1}(y) - \frac{y}{c} \right| .$$

Here function $h(x)$ is sub-additive.

Definition. Distribution F^s with an unbounded support is *intermediate regularly varying* (IRV) if

(2) $$\lim_{c\downarrow 1} \lim \inf_{x\to\infty} \overline{F}^s(cx)/\overline{F}^s(x) = 1.$$

IRV distributions form a subclass of subexponential distributions which includes, in particular, all regularly varying distributions. It is known that a distribution is IRV if and only if it is h-insensitive, for any function $h(x) = o(x)$. So, for an IRV distribution F^s, (IIa) implies (IIb).

Here are two results for asymptotically proportional insurance/reinsurance, first for IRV distributions and then for a class of distributions with lighter tails.

Theorem. Assume that distribution F^s is IRV. Let conditions (I) and (IIa) to hold. Assume $1/c \geq 1/\mu(1-c)$.

(1) If, in addition, $a_I/c \geq a_R/(1-c)$, then

$$\psi_{\mathrm{I}\wedge\mathrm{R}}(u,v) \sim \psi_{\mathrm{I}=\mathrm{R}}(u,v) \sim \psi_{\mathrm{I}}(u) \sim \frac{c}{a_I}\overline{F}^s(u/c).$$

(2) If, on the contrary, $a_I/c < a_R/(1-c)$, then

$$\begin{aligned}
\psi_{\mathrm{I}\wedge\mathrm{R}}(u,v) \quad &\sim \quad \psi_{\mathrm{I}=\mathrm{R}}(u,v) \sim \psi_{\mathrm{I}}(u) - \psi_{\mathrm{I}}(u(1+K)) + \psi_{\mathrm{R}}(u(1+K)) \\
&\sim \quad \frac{c}{a_I}\left(\overline{F}^s(u/c) - \overline{F}^s(u(1+K)/c)\right) + \frac{\mu(1-c)}{a_R}\overline{F}^s(u(1+K)/c)
\end{aligned}$$

where

$$K = \left(\frac{1}{c} - \frac{1}{\mu(1-c)}\right)\left(\frac{a_R}{1-c} - \frac{a_I}{c}\right)^{-1}.$$

Here is a general result for subexponential distributions that are not IRV.

Theorem. Assume that the distribution F^s is subexponential and that

(3) $$\lim_{x\to\infty}\sup \overline{F}^s(kx)/\overline{F}^s(x) = 0,$$

for all $k > 1$. Assume further conditions (I), (IIa) and (IIb) to hold. If

$$1/c > 1/\mu(1-c),$$

then

$$\psi_{\mathrm{I}\wedge\mathrm{R}}(u,v) \sim \psi_{\mathrm{I}=\mathrm{R}}(u,v) \sim \psi_{\mathrm{I}}(u).$$

By the symmetry, if

$$1/c < 1/\mu(1-c),$$

then

$$\psi_{\mathrm{I}\wedge\mathrm{R}}(u,v) \sim \psi_{\mathrm{I}=\mathrm{R}}(u,v) \sim \psi_{\mathrm{R}}(v).$$

Remark. We have left out the boundary case $1/c = 1/\mu(1-c)$ because here we may get a variety of individual asymptotics that depend on the tail asymptotics of distribution F. The result is the same as in the latter theorem if, in addition, F^s is $|h|$-insensitive, with $h(u) = g(u)/(1-c) - u/c$ (here $g(u) = v$). However, if $h(u)/u$ tends to 0 relatively slowly, the asymptotics may differ.

Consider the following example. Let $\overline{F}^s(x) = e^{-\sqrt{x}}$ and assume that $h(u) = u^\alpha \sin u$ with $\alpha \in (1/2, 1)$.

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