Recent Results in Heavy-tailed and Dependent Risk Theory

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Outline of Talk:

1. Review of Classical Ruin Model
2. Review of Light-tailed/Heavy-tailed Theory
3. Accuracy of Light-tailed/Heavy-tailed Approximations
4. Improving the Heavy-tailed Approximation
5. Simulation in the Presence of Heavy Tails
6. Heavy-tailed Modeling: a Common Complaint
7. Remarks on the Dependent Case

The Basic Risk Model

$$R(t) = x + ct - \sum_{i=1}^{N(t)} V_i$$

Problem 1: What is the probability of eventual ruin? (similar questions for finite horizon)

$$\tau = \inf\{t \geq 0 : R(t) \leq 0\}$$

$$\mathbb{P}_x(\tau < \infty) = ?$$

Problem 2: What is the "excess over the boundary"? (related to global reinsurance based on retention limit x)

$$\mathbb{P}_x(R(t) \in \cdot | \tau < \infty) = ?$$

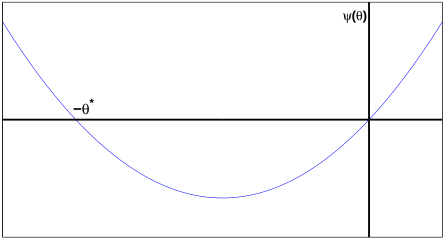
Note that:

$$R(T_n) = \sum_{i=1}^{n} (c\chi_i - V_i)$$

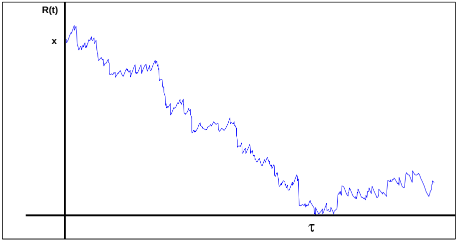
Assume $((\chi_i, V_i) : i \geq 1)$ iid

Light-tailed Review

- tail of V_i decays (at least) exponentially rapidly
- $\psi(\theta) = \log \mathbb{E} \exp(\theta(c\chi_1 - V_1))$



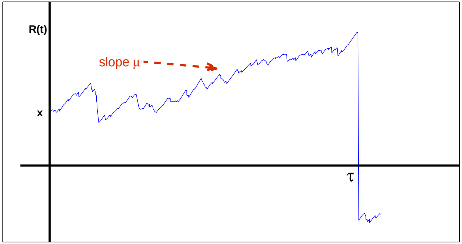
- $\mathbb{P}_x(\tau < \infty) \sim ce^{-\theta^* x}$ as $x \to \infty$ (Cramér-Lundberg)



- $\mathbb{P}_x((\chi_i, V_i) \in A_i, 1 \leq i \leq n | \tau < \infty) \Rightarrow \prod_{i=1}^{n} \mathbb{E} I((\chi_1, V_1) \in A_i) \exp(-\theta^*(c\chi_1 - V_1))$ as $x \to \infty$
- excess over boundary $= O(1)$

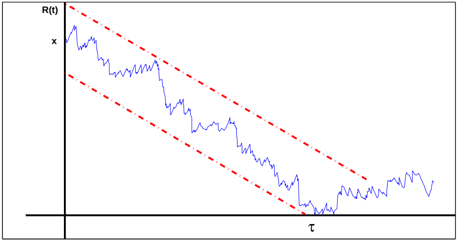
Heavy-tailed Review

- $\mathbb{P}(V > x) \sim cx^{-\alpha}$ as $x \to \infty$
- $\mathbb{P}(V > x) \sim c\exp(-(\lambda x)^{\alpha})$ as $x \to \infty$, $0 < \alpha < 1$



- $\mathbb{P}_x(\tau < \infty) \sim \int_0^\infty \mathbb{P}(V_1 > x + \mu s)\mathrm{d}s$ as $x \to \infty$ (Pakes, Embrechts and Veraverbeke)
- $\mathbb{P}_x(\tau \in \mathrm{d}t) \approx \frac{\mathbb{P}(V_1 > x + \mu t)\mathrm{d}t}{\int_0^\infty \mathbb{P}(V_1 > x + \mu s)\mathrm{d}s}$
- excess over boundary $= O(x)$

Accuracy of Light-tailed Approximations



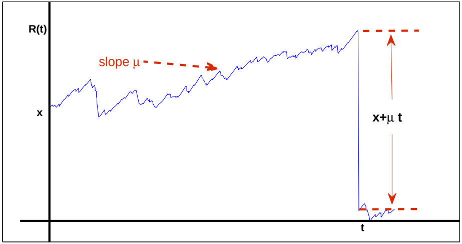
- rare event path exponentially more likely than competing paths to ruin
- relative accuracy

$$\frac{\mathbb{P}_x(\tau < \infty)}{c e^{-\theta^* x}} - 1 \to 0$$

exponentially rapidly

Accuracy of Heavy-tailed Approximations

Competing rare-event path:



- probability of such a path is roughly $O(x^{-\alpha-1})$
- integrate over t: error of order $x^{-\alpha}$
- relative error $= O(x^{-1})$
- not very accurate approximation

Improving the Heavy-tailed Approximation

- We consider the regime in which the "safety loading" is close to zero...
- This is one in which the reserve process can be approximated by a Brownian motion

Diffusion Approximation

Define $\sigma^2 \triangleq \mathrm{var}(V_1 - \tilde{c}\chi_1) < \infty$, where $\tilde{c} = \frac{\mathbb{E}V_1}{\mathbb{E}\chi_1}$.
If μ is close to 0, then

$$\mathbb{P}_x(\tau < \infty) \approx \exp\left(-\frac{2\mu x}{\sigma^2}\right)$$
$$R(t) \overset{\mathscr{D}}{\approx} x + \mu t + \sigma B(t)$$

over spatial scales of order $\frac{1}{\mu}$ and temporal scales of order $\frac{1}{\mu^2}$. So, if $x = \Theta(\frac{1}{\mu})$, then $\mathbb{P}_x(\tau < \infty) \approx \exp\left(-\frac{2\mu x}{\sigma^2}\right)$.
Contrast this with Cramér-Lundberg approximation:

$$\mathbb{P}_x(\tau < \infty) \sim c(\mu) \exp(-\theta^*(\mu) x) \text{ as } x \to \infty$$

Clearly, it must be that:

$$c(\mu) \to 1 \text{ as } \mu \to 0$$
$$\theta^*(\mu) \sim \frac{2\mu}{\sigma^2} \text{ as } \mu \to 0$$

Corrected Diffusion Approximations (CDAs)

Both $c(\cdot)$ and $\theta^*(\cdot)$ can be expanded in power series in μ:

$$\theta^*(\cdot) = \frac{2\mu}{\sigma^2} + a_2\mu^2 + a_3\mu^3 + \cdots$$
$$c(\cdot) = \exp(c_1\mu + c_2\mu^2 + \cdots)$$

where a_k depends on first $k+1$ moments of V_i's and χ_i's.

This suggests that we can get an approximation accurate to order $o(\mu^r)$ when x is of order $(\frac{1}{\mu})^l$ (by suitably expanding c to order r and θ^* to order $r+l$).

Also, these approximations make formal sense even in the heavy-tailed setting, provided that enough moments are finite

Back to Heavy Tails...

We know that when x is very big that the heavy-tail approximation must hold...

And when x is of the scale described by the diffusion approximation, it must hold...

Where do the two approximations cross?

In the regularly varying case in which $\mathbb{P}(V > x) \sim L(x)x^{-\alpha}$ as $x \to \infty$, the critical threshold is at:

$$x^* = \frac{1}{\mu} \log\left(\frac{1}{\mu}\right) \frac{\mathbb{E}V^2}{2\mathbb{E}V}(\alpha - 2)$$

when the χ_i's form a Poisson process having unit rate

Theorem (Olvera + G(2010))

If $\limsup \frac{x}{x^*} < 1$, then
$\mathbb{P}_x(\tau < \infty) \sim$ CDA approximation as $x \to \infty$ and $\mu \to 0$
whereas if $\liminf \frac{x}{x^*} > 1$, then
$\mathbb{P}_x(\tau < \infty) \sim$ heavy tail approximation as $x \to \infty$ and $\mu \to 0$

Weibull case...

Same theorem as in regularly varying setting, except

$$x^* = d(\alpha)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\alpha}}$$

when the χ's from a Poisson process with unit rate and $\mathbb{P}(V > x) \sim c\exp(-(\lambda x)^\alpha)$ as $x \to \infty$

Bottom line:

Numerical simulations indicate that these blended approximations are much better than either one by itself

Simulation in the Presence of Heavy Tails

If the claims process is simulated by crude Monte Carlo, two problems arise:

1. The simulation may not terminate, because we may be simulating a path on which ruin does not occur
2. The number of paths needed to generate even a single ruin grows as $x \to \infty$

Importance Sampling:
Simulate the process using a different law that induces ruin more frequently and use the likelihood ratio along the simulated path to produce an estimator appropriate to the original dynamics

$$I(\tau < \infty) \prod_{i=1}^{\tau} r(R(T_{i-1}), R(T_i))$$

Importance Sampling

Ideal law under which to simulate: Conditional distribution given ruin

$$\tilde{\mathbb{P}}(y, \mathrm{d}z) = \mathbb{P}(y, \mathrm{d}z)\frac{u^*(z)}{u^*(y)}$$

where $u^*(y) = \mathbb{P}_y(\tau < \infty)$

We don't know $u^*(\cdot)$ but we do know a good asymptotic for it...

i.e. $u^*(y)$ is asymptotic to $v(y)$ as $y \to \infty$

So, use:

$$\tilde{\mathbb{P}}(y, \mathrm{d}z) = \mathbb{P}(y, \mathrm{d}z)\frac{v(z)}{w(y)} \quad \text{(Blanchet+G(08))}$$

where $w(\cdot)$ is the normalization factor

This works (and is asymptotically optimal in some sense). (Proof is based on Lyapunov function ideas)

Heavy-tailed Modeling: A Common Complaint

- Heavy-tail asymptotics in ruin theory require that the claims be sub-exponential (e.g. regularly varying) "all the way to infinity"

- The tail is essentially modeled parametrically

- How can one verify this statistically from a finite collection of observed data?

Suppose that one can accurately estimate $\mathbb{E}V$ and suppose one has a good estimate of $\mathbb{P}(V > x)$ over $x \in [0, y]$, where $\mathbb{P}(V > \cdot)$ is regularly varying.

Then, compute ruin probability for $\mathbb{P}_x(\tilde{\tau} < \infty)$ based on regularly varying claims distribution $\tilde{V}$ that matches:

$$\mathbb{E}V$$

and

$$\mathbb{P}(V > x) \text{ over } x \in [0, y]$$

Theorem (Olvera + G(2007))

$\mathbb{P}_w(\tau < \infty) \sim \mathbb{P}_w(\tilde{\tau} < \infty)$ provided $y/w \to \infty$

The Dependent Case

- results depend critically on the specific type of dependence induced
- For auto-regressive type dependence:

$$V_n = \beta V_{n-1} + \epsilon_n \quad 0 < \beta < 1$$

  with $\mathbb{P}(\epsilon_n > z) \sim cz^{-\alpha}$ as $z \to \infty$, ϵ_n's iid
- If V_n is big ($\approx \gamma$), then $V_n + k \approx \gamma\beta^k$ so the total claims from n to $n + l$ are:

$$\sum_{k=n}^{n+l} V_k \approx \gamma \sum_{k=0}^{l} \beta^k \approx \frac{\gamma}{1-\beta}$$

  if l is large
- can effectively be treated as if one large claim of size $\gamma(1-\beta)^{-1}$ arrived at T_n
- can have big impact on both global and local reinsurance

Long-range Dependence

$$R(t) = x + \mu t - \Gamma(t)$$

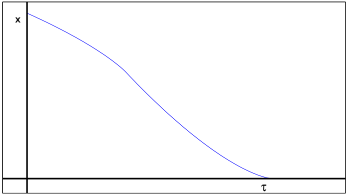
- Γ fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$
- stationary increments
- Gaussian with continuous paths
- $\mathrm{corr}(\Gamma(t) - \Gamma(t-1), \Gamma(1) - \Gamma(0)) \sim ct^{2H-2}$ as $t \to \infty$
- arises as limit for insurance in which a given claim payment occurs not as a lump sum but as a stream that persists over long time scales

Hüsler and Piterbarg (1999):

$$\mathbb{P}_x(\tau < \infty) \sim C\sqrt{\frac{\pi}{H}}2^{\frac{2H}{1-H}}A^{1-H}(1-H)^{\frac{2H}{2-H}}x^{\frac{(1-H)^2}{H}}(1-\Phi(\frac{A^H x^{1-H}}{1-H}))$$

as $x \to \infty$, where $A = \frac{\mu(1-H)}{H}$ and C is the Pickands constant defined by $C \triangleq \lim_{T\to\infty} T^{-1}\mathbb{E}\exp\left(\sup_{0\le t\le T}(\sqrt{2}\Gamma(t) - t^{2H})\right)$

Chang, Yao, and Zajic (1999):



Remark: Scaling law that describes microscopic behavior of path in neighborhood of tau is also available (Awad and G (09))

Conclusions

- Improved approximations for heavy-tailed systems
- Simulation of heavy-tailed systems
- Impact of dependence