Comonotonicity of Asset Prices in Arbitrage-Free Markets

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A set $A \subset R^n$ is called **comonotonic** if for any x and y in A, either $x \le y$ or $y \le x$ holds.

Here $\le$ denotes the component-wise order.

A random vector X is called **comonotonic** if it has a comonotonic support.

A random variable ξ has an isolated mass at point z if $\Pr(\xi = z) > 0$ and there exists a δ such that $\Pr(z - \delta < \xi < z + \delta, \xi \ne z) = 0.$

Consider a **financial market consisting of a stock** and a bank account. One can deposit money on or borrow money from the bank at constant interest rate $r > 0.$
The price of the stock at time t is given by the random vector $S_t.$

Theorem 1. *Suppose the market is arbitrage-free. Consider the moments* $0 < u < v.$ *Assume additionally that for any isolated mass point z of* $S_u,$
$\Pr(S_u = z, S_v < S_u e^{r(v-u)}) \times \Pr(S_u = z, S_v > S_u e^{r(v-u)}) = 0.$
In this case, the following two statements are equivalent:

(1) *The vector* (S_u, S_v) *is comonotonic.*
(2) The couple is connected by the linear relationship $S_v = S_u e^{r(v-u)},$ *a.s.*

Consider a **financial market consisting of** $n \ge 2$ **stocks** and a bank account with constant interest rate $r > 0$. The prices of the stocks, labeled from 1 to *n*, at a fixed moment *t* are given by the random variables $S^i(t)$.

Let *W(t)* be a Wiener process and

$$\frac{dS^i(t)}{S^i(t)} = \mu_i dt + \sigma_i dW(t), \quad i = 1,\dots,n.$$

Here σ_i are positive and $S^i(0)$ are positive and nonrandom.

The price vector $(S^1(t), S^2(t),\dots,S^n(t))$ is always *comonotonic*.

Theorem 2. *This market is arbitrage-free if, and only if,*

$$\mu_i = r + a\sigma_i, \; i = 1,\dots,n,$$

for some fixed real number a.

Consider a more **general multivariate B&S market**

$$\frac{dS^i(t)}{S^i(t)} = \mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW^j(t), \quad i = 1,\dots,n.$$

Here $(W^1(t),\dots,W^d(t))$ is a standard d-dimensional Brownian motion. The initial stock prices are positive and nonrandom.
Suppose that the underlying filtration coincides with one generated by the Brownian motion.
Let Σ be the $n \times d$ volatility matrix with entries σ_{ij}.

Theorem 3. The following two statements are equivalent:

(1) The B&S market is complete.
(2) $d \le m$ and the matrix Σ has full rank, d.

Let μ be a column vector with components μ_i.

Theorem 4. *The following statements are equivalent.*

(1) The market is arbitrage-free.
(2) There exists an equivalent martingale measure.
(3) $\mu - r\boldsymbol{1} \in R(\Sigma)$.

Corollary 5. *Suppose that* $\mu - r\boldsymbol{1} \in R(\Sigma)$. *Then the market is arbitrage-free, and there exists an equivalent martingale measure. Moreover, the following statements are equivalent:*

(1) The market is complete.
(2) $d \le m$ and the matrix Σ has full rank, d.
(3) The equivalent martingale measure is unique.

Now, suppose additionally that **dividends** D_t are paid to the stock holder according to the formula

$$dD_t = k\gamma_t^T S_t dt.$$

Here $k>0$ is fixed rate, S_t is the price vector, and $\gamma_t^T S_t$ is the price of the risky part of the portfolio.

Then the market is equivalent to the former market without dividends, where μ_i are replaced by $\mu_i + k.$

And Theorems 3 and 4 are applicable.

Example of comonotonic arbitrage-free market with given marginals

Consider the market (B, S^1, S^2) consisting of bonds with interest rate *r>0* and two stocks. For *T>0* suppose that the price vector at moment *T* is comonotonic with

$$S_T^i = S_0^i \exp(\nu_i T + \sigma_i \tau), \quad i = 1, 2.$$

Here S_0^i are initial prices, $0 < \sigma_1 < \sigma_2$, and the random variable τ has a strictly monotone cdf *F(x)* on real line.

The market is considered only for two discrete moments *t=0* and *t=T*.

Theorem 6. The market is arbitrage-free if, and only if,

$$\frac{\nu_1 - r}{\sigma_1} > \frac{\nu_2 - r}{\sigma_2}.$$

Here the idea of existence of equivalent martingale measure is not applicable.

Corollary 7. Assume additionally that

$\nu_i = \mu_i - \frac{\sigma_i^2}{2}, i = 1, 2,$ and $\tau \sim N(0, T).$

(Like for B&S market).

Then the market is arbitrage-free if, and only if,

$$\frac{\mu_1 - r}{\sigma_1} + \frac{\sigma_2 - \sigma_1}{2} > \frac{\mu_2 - r}{\sigma_2}.$$