COMPOUND BIRTH PROCESSES IN RISK MODELS

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Abstract

In this note a stochastic counting process is defined as a compound birth process. We consider the examples of Pólya - Aeppli process, I - Pólya process and I - Binomial process. The insurance risk model in which the counting process is a compound birth process is also defined. Some basic properties are given. The differential equation for the joint probability distribution of the time to ruin and the deficit at ruin is derived. The case of exponentially distributed claims is discussed.

Key words: Birth process, Pólya - Aeppli process, ruin probability, I - Pólya process.

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1 Definition

Let $\{N(t), t \geq 0\}$ be the number of times a certain event occurs in time interval $(0, t]$ and let $\{p_i,\ i = 1, 2, \ldots\}$ be the distribution of some compounding random variable with values in $\{1, 2, \ldots\}$. The transition probabilities of the counting process $N(t)$, for every $m = 0, 1, \ldots$ are defined by the following postulates:

$$P(N(t+h) = n \mid N(t) = m) = \begin{cases} 1 - \sum_{i=1}^{\infty} p_i \lambda_i(t) h + o(h), & n = m, \\ p_i \lambda_i(t) h + o(h), & n = m + k, k = 1, 2, \ldots, \end{cases}$$

where $o(h) \to 0$ as $h \to 0$ and $\lambda_i(t),\ i = 1, 2, \ldots$ are intensity functions.

Let $A(t) = \sum_{i=1}^{\infty} p_i \lambda_i(t)$. If $P_m(t) = P(N(t) = m),\ m = 0, 1, 2, \ldots$, the above postulates yield the following Kolmogorov forward equations:

$$P_0'(t) = -A(t) P_0(t),$$

(1)
$$P_m'(t) = -A(t) P_m(t) + \sum_{i=1}^{m} p_i \lambda_i(t) P_{m-k}(t),\ m = 1, 2, \ldots.$$

This leads to the following definition

Definition 1 *The counting process $N(t)$, defined by the differential equations (1) with initial conditions*

$$P_0(0) = 1 \quad \textit{and} \quad P_m(0) = 0, \quad m = 1, 2, \ldots$$

is called a compound birth process.

The compound birth process extends several well known processes.

2 Probability generating function

Multiplying the mth equation of (1) by u^m and summing for all $m = 0, 1, 2, \ldots$ we get the following differential equation for the probability generating function

$$\frac{\partial P_{N(t)}(s)}{\partial t} = -\sum_{i=1}^{\infty} p_i \lambda_i(t)(1 - s^i) P_{N(t)}(s). \tag{2}$$

The solution of (2) with the initial condition $P_{N(t)}(0) = 1$ is given by

$$P_{N(t)}(s) = \exp\left(-\sum_{i=1}^{\infty} p_i (1 - s^i) \int_0^t \lambda_i(u) du\right). \tag{3}$$

3 Application to Risk Theory

Consider the standard risk model $\{X(t),\ t \geq 0\}$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and given by

$$X(t) = ct - \sum_{k=1}^{N(t)} Z_k, \quad \left(\sum_1^0 = 0\right). \tag{4}$$

Here c is a positive real constant representing the risk premium rate. The sequence $\{Z_k\}_{k=1}^{\infty}$ of non-negative mutually independent identically distributed random variables is independent of the counting process $N(t),\ t \geq 0$. The claim sizes $\{Z_k\}_{k=1}^{\infty}$ are distributed as the random variable Z with distribution function $F,\ F(0) = 0$ and mean value μ.

We consider the risk model (4), where $N(t)$ is a compound birth process.

The relative safety loading θ is defined by

$$\theta = \frac{ct}{\mu \sum_{i=1}^{\infty} i p_i \int_0^t \lambda_i(s)} - 1,$$

and in the case of positive safety loading $\theta > 0, \quad c > \frac{\mu}{t} \sum_{i=1}^{\infty} i p_i \int_0^t \lambda_i(s) ds$.

Let $\tau = \inf\{t : X(t) < -u\}$ with the convention of $\inf \emptyset = \infty$ be the time to ruin of an insurance company having initial capital $u \geq 0$. We denote by $\Psi(u) = P(\tau < \infty)$ the ruin probability and $\Phi(u) = 1 - \Psi(u)$ the non-ruin probability.

In the following we use the notation of [2]. Let $G(u, y)$ be the joint probability distribution of the time to ruin τ and the deficit at the time of ruin D, i.e.

$$G(u, y) = P(\tau \leq \infty, D \leq y). \tag{5}$$

and

$$\lim_{y \longrightarrow \infty} G(u, y) = \Psi(u). \tag{6}$$

Using the postulates we have

$$G(u,y) = (1 - A(t)h)\, G(u+ch,y) +$$

$$+\sum_{i=1}^{\infty} p_i \lambda_i(t) h \left[\int_0^{u+ch} G(u+ch-x,y)dF^{\star i}(x) + F^{\star i}(u+ch+y) - F^{\star i}(u+ch) \right] + o(h),$$

where $F^{\star i}(x), \quad i = 1, 2, \ldots$ is the distribution function of $Z_1 + Z_2 + \ldots + Z_i$.
Rearranging the terms leads to

$$\frac{G(u+ch,y) - G(u,y)}{ch} = \frac{A(t)}{c} G(u+ch,y) -$$

$$-\frac{1}{c}\sum_{i=1}^{\infty} p_i \lambda_i(t) \left[\int_0^{u+ch} G(u+ch-x,y)dF^{\star i}(x) + F^{\star i}(u+ch+y) - F^{\star i}(u+ch) \right] + \frac{o(h)}{h},$$

Let

$$H(x) = \sum_{i=1}^{\infty} p_i \lambda_i(t) F^{\star i}(x)$$

be the defective probability distribution function of the claims with

$$H(0) = 0 \ \text{ and } \ H(\infty) = A(t).$$

By letting $h \to 0$ we obtain the following differential equation

$$\frac{\partial G(u,y)}{\partial u} = \frac{A(t)}{c} G(u,y) - \frac{1}{c} \left[\int_0^u G(u-x,y)dH(x) + [H(u+y) - H(u)] \right]. \tag{7}$$

In terms of the proper probability distribution function $H_1(x) = \frac{H(x)}{A(t)}$ the equation (7) is given by

$$\frac{\partial G(u,y)}{\partial u} = \frac{A(t)}{c} \left[G(u,y) - \int_0^u G(u-x,y)dH_1(x) + [H_1(u+y) - H_1(u)] \right]. \tag{8}$$

Theorem 1 *The function $G(0,y)$ is given by*

$$G(0,y) = \frac{A(t)}{c} \int_0^y [1 - H_1(u)]du. \tag{9}$$

Proof. Integrating (8) from 0 to ∞ with $G(\infty, y) = 0$ leads to

$$-G(0,y) =$$

$$= \frac{A(t)}{c} \left[\int_0^{\infty} G(u,y)du - \int_0^{\infty}\int_0^u G(u-x,y)dH_1(x)du - \int_0^{\infty} (H_1(u+y) - H_1(u))du \right]$$

The change of variables in the double integral and simple calculations yield

$$G(0,y) = \frac{A(t)}{c} \int_0^{\infty} [H_1(u+y) - H_1(u)]du$$

and (9).

□

3.1 Ruin probability

Theorem 2 *The probability of ruin $\Psi(u)$ satisfies the equation*

$$\frac{\partial \Psi(u)}{\partial u} = \frac{A(t)}{c}\left[\Psi(u) - \int_0^u \Psi(u-x)dH_1(x) + [1-H_1(u)]\right], \quad u \geq 0. \tag{10}$$

Proof. The result follows from (8) and (6).

□

Theorem 3 *The ruin probability with no initial capital satisfies*

$$\Psi(0) = \frac{\mu}{c}[p_1\lambda_1(t) + 2p_2\lambda_2(t) + 3p_3\lambda_3(t) + \ldots]. \tag{11}$$

Proof. According (6) and (9)

$$\Psi(0) = \lim_{y\to\infty} G(0,y) = \frac{A(t)}{c}\int_0^\infty [1-H_1(u)]du.$$

Let X be a random variable with distribution function $H_1(x)$. By the definition of $H_1(x)$ and $EZ = \mu$ we obtain

$$EX = \frac{\mu}{A(t)}[p_1\lambda_1(t) + 2p_2\lambda_2(t) + 3p_3\lambda_3(t) + \ldots].$$

Using the fact that $EX = \int_0^\infty [1-H_1(x)]dx$ we obtain (11).

□

4 Examples

Consider the case of geometric compounding distribution with parameter $\rho \in [0,1)$, i.e.

$$p_i = (1-\rho)\rho^{i-1}, \quad i = 1, 2, \ldots.$$

A special choice of the intensity functions leads to the counting processes defined in [4] with distributions of [5]. The good properties of the defined processes are explained by the lack of memory property of the geometric distribution.

4.1 Pólya - Aeppli process

In the case of constant intensity functions:

$$\lambda_i(t) = \lambda, \quad i = 1, 2, \ldots$$

the function $A(t)$ is also a constant:

$$A(t) = \lambda.$$

The solution of the equations (1) is given by the Inflated-parameter Poisson distribution, [5]:

$$P(N(t) = n) = \begin{cases} e^{-\lambda t}, & n = 0 \\ e^{-\lambda t}\sum_{i=1}^n \binom{n-1}{i-1}\frac{[\lambda(1-\rho)t]^i}{i!}\rho^{n-i}, & n = 1, 2, \ldots. \end{cases} \tag{12}$$

The Inflated-parameter Poisson distribution coincides with the Pólya- Aeppli distribution, see [1].

Definition 2 *([6]) The counting process, defined by (12) is called a Pólya - Aeppli process.*

It is easy to prove also that the Pólya - Aeppli process is the only compound Poisson process, that is a renewal process. The time T_1 until the first claim epoch and the inter - arrival times $T_2, T_3, \ldots$ are nonnegative, mutually independent random variables. T_1 is exponentially distributed with parameter λ. $T_2, T_3, \ldots$ are identically distributed as a random variable T_2. Moreover T_2 is zero with probability ρ, and with probability $1-\rho$ exponentially distributed with parameter λ, i.e. T_2 is exponentially distributed with mass at zero equal to ρ. The probability distribution function is given by

$$F_{T_2}(t) = 1 - (1-\rho)e^{-\lambda t},\ t \geq 0.$$

The mean value is $ET_2 = \frac{1-\rho}{\lambda}$.

4.2 I - Pólya process

Consider for some $r \geq 1$ and $\beta > 0$, the intensity functions:

$$\lambda_i(t) = \frac{r}{\beta}\left(\frac{\beta}{\beta+t}\right)^{i+1}\left[1+\frac{t}{\beta\rho}\right]^{i-1},\quad \lambda_i(0) = \frac{r}{\beta},\quad i = 1, 2, \ldots.$$

The function $A(t)$ is given by

$$A(t) = \frac{r}{\beta+t}.$$

In this case the solution of the equations (1) gives the following probability mass function:

$$P(N_t = m) = \begin{cases} \left(\frac{\beta}{\beta+t}\right)^r, & m = 0 \\ \left(\frac{\beta}{\beta+t}\right)^r \sum_{i=1}^{m}\binom{m-1}{i-1}\binom{r+i-1}{i}[(1-\rho)\frac{t}{\beta+t}]^i\rho^{m-i}, & m = 1, 2, \ldots. \end{cases} \tag{13}$$

This is the Inflated-parameter Negative binomial distribution with parameters $\frac{\beta}{\beta+t}$, ρ and r ([3] and [5]). We use the notation $N(t) \sim INB\left(\frac{\beta}{\beta+t}, \rho, r\right)$.

Definition 3 *The counting process, $\{N(t),\ t \geq 0\}$ is said to be an Inflated - parameter Pólya process or I - Pólya process, if it starts at zero, $N(0) = 0$ and for each $t > 0$, the distribution of $N(t)$ is given by (13).*

It is easy to find that the time to the first claim has Pareto distribution

$$F_{T_1}(t) = 1 - \left(\frac{\beta}{\beta+t}\right)^r,\quad t \geq 0. \tag{14}$$

and the distribution function of the claims is

$$H_1(x) = (1-\rho)\frac{\beta}{\beta+t}\sum_{i=1}^{\infty}\left[1-(1-\rho)\frac{\beta}{\beta+t}\right]^{i-1}F^{*i}(x).$$

4.3 I - Binomial process

In this case, for some $\alpha > 0$, the intensity functions are given by

$$\lambda_i(t) = \frac{n}{\alpha}\left(\frac{\alpha}{\alpha - t}\right)^{i+1}\left[1 - \frac{t}{\alpha\rho}\right]^{i-1}, \; t < \alpha, \quad i = 1, 2, \dots$$

and

$$A(t) = \frac{n}{\alpha - t}.$$

The solution of the equations (1) is given by the Inflated-parameter binomial distribution with parameters $\frac{t}{\alpha}$, ρ and n ([3] and [5]):

$$(15) \qquad P(N_t = m) = \begin{cases} \left(1 - \frac{t}{\alpha}\right)^n, & m = 0 \\ \sum_{i=1}^{m \wedge n} \binom{n}{i}\binom{m-1}{i-1}\left[(1-\rho)\frac{t}{\alpha}\right]^i \left(1 - \frac{t}{\alpha}\right)^{n-i} \rho^{m-i}, & m = 1, 2, \dots. \end{cases}$$

The notation is

$$N(t) \sim IBi\left(\frac{t}{\alpha}, \rho, n\right).$$

Definition 4 *The counting process $\{N(t), \; t \geq 0\}$ is said to be I - Binomial process, if it starts at zero, $N(0) = 0$ and for each $t > 0$, the distribution of $N(t)$ is given by (15).*

For the I-Binomial process the distribution function of the time to the first claim is

$$(16) \qquad F_{T_1}(t) = 1 - \left(1 - \frac{t}{\alpha}\right)^n, \quad t < \alpha.$$

and the distribution function of the claims

$$H_1(x) = (1-\rho)\frac{\alpha}{\alpha - t}\sum_{i=1}^{\infty}\left[1 - (1-\rho)\frac{\alpha}{\alpha - t}\right]^{i-1} F^{*i}(x).$$

4.4 Exponentially distributed claims

Let us consider the case of exponentially distributed claim sizes, i.e. $F(u) = 1 - e^{-\frac{u}{\mu}}, \; u \geq 0, \; \mu > 0$. In this case, the probability of ruin for the defined risk model is

$$\Psi(u) = \frac{1}{1+\theta} e^{-\frac{A(t)}{c}\theta u},$$

where the function $A(t)$ and the relative safety loading coefficient θ are defined by the corresponding counting processes.

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