The Gerber-Shiu Penalty Function for a Risk Process with Two Classes of Claims under a Multi-layer Dividend Strategy

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Abstract

In this paper we consider a risk model with two classes of insurance risks in the presence of a multi-layer dividend startegy. We assume that the two claim counting processes are, respectively, Poisson and Sparre Andersen with generalized Erlang(2) claim inter-arrival times. We derive an integro-differential equation system for the Gerber-Shiu functions for surplus-dependent premium rates and as a special case we obtain an integro-differential equation system for the multi-layer risk model. The solution of the above system is given in terms of the linearly independent solutions to the associated homogeneous integro-differential equation system. The analysis of this homogeneous integro-differential systems is considered using Laplace transforms. Then we provide a recursive approach to obtain the general solution of the Gerber-Shiu functions.

Keywords. Compound Poisson process, Generalized Erlang risk process, Defective renewal equations, Multi-layer dividend strategy, Intergro-differential equations, Rationally distributed claim severities.

1 Introduction

Dividend strategies for insurance risk models were first proposed by De Finetti (1957) to describe more realistically the surplus cash flows in insurance portfolios. Most of the dividend strategies are of the following two kinds: one is the constant barrier strategy and another is the threshold strategy. Strategies involving a single horizontal barrier have been studied by Lin et al. (2003) for the classical compound Poisson risk model and by Li and Garrido (2004b) for the renewal generalized Erlang(n) risk model. On the other hand, strategies involving a single dividend threshold have been studied by Dickson and Drekic (2006), Lin and Pavlova (2007) for the classical compound Poisson risk model and by Albrecher et al. (2007) for the renewal generalized Erlang(n) risk model.

Recently, the multi-layer dividend strategy, as an extension of the threshold dividend strategy has been investigated in several papers. Zhou (2007), Lin and Sendova (2007), Albrecher and Hartinger (2007) considered a multiple-layer setting within the framework of the classical risk model, while Yang and Zhang (2007) investigated the multi-layer strategy in the renewal generalized Erlang(n) risk model. The central focus of these papers is the characterization of the expected discounted penalty function, introduced by Gerber and Shiu (1998), (see also Gerber and Shiu (2005)).

In recent years, many authors have studied various aspects of the so-called correlated aggregate claims risk model. Yen et al. (2002) introduced a correlated risk process involving two dependent classes of insurance risks, which can be transformed into a surplus process with two independent classes of insurance risks, for which one claim number process is Poisson and the other is a renewal process with Erlang(2) claim inter-arrival times. Li and Garrido (2005) consider a risk process with two classes of independent

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risks, namely the compound Poisson and the compound renewal with generalized Erlang (2) inter-arrival times and they derived explicit results for the non-ruin probability. Li and Lu (2005) extended the results of Li and Garrido (2005) by studying the Gerber-Shiu expected discounted penalty functions, when ruin is caused by a claim belonging either to the first or the second class. A further investigation was made by Zhang et al. (2009) by studying the Gerber-Shiu functions of Li and Lu (2005) when the claim number process of the second class is a renewal process with generalized Erlang(n) inter-arrival times. Recently, Chadjiconstantinidis and Papaioannou (2009) consider the model of Zhang et al. (2009) and they study the Gerber-Shiu functions in the presence of a constant dividend barrier strategy as well as the moments of the discounted sum of the dividend payments until ruin. In the same paper a dividends-penalty identity is derived. Finally Lu et al. (2009) extended the results of Li and Lu (2005) in the case where a threshold dividend strategy is applied.

Motivated by the study of the compound Poisson and compound renewal model with multi-layer dividend strategy, we investigate some corresponding results in a risk model with two classes of claims in which the two claim number processes are independent Poisson and generalized Erlang(2) processes, respectively.

First we start by introducing a general risk model with two classes of claims, in which the premium rate is surplus dependent. Let $U(t)$ be the surplus process at time t with $u = U(0)$, satisfying

$$dU(t) = c(U(t))dt - dS(t), \qquad t \geq 0, \tag{1}$$

where $c(U(t))$ is the premium rate at time t, and $c(x)$ is a positive deterministic function such that $\int_0^t [c(x)]^{-1} dx < \infty$, for finite number $t \geq 0$ and $\int_0^\infty [c(x)]^{-1} dx = \infty$. We assume that $S(t)$ is generated by two classes of insurance risks, namely

$$S(t) = S_1(t) + S_2(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0, \tag{2}$$

where $S_i(t)$, $i = 1, 2$, represents the aggregate claims up to time t from the i-th class. Although such models are usually studied in the context of correlated aggregate claims, here we assume that $S_1(t)$ and $S_2(t)$ are stochastically independent. Under the independence of $S_1(t)$ and $S_2(t)$ we may say that we look at the impact on the surplus of an extra variability. The aggregate 'shocks' (or claims) $S_1(t)$, from the first class, is added independently to the natural random variability of the aggregate insurance claims $S_2(t)$, from the second class as in Li and Garrido (2005).

The random variables (r.v.) $\{X_i\}_{i=1}^\infty$ are the positive claim severities from the first class, which are independent and identically distributed (i.i.d.) r.v. with common distribution function (d.f.) $F_1(x) = \mathbb{P}(X \leq x)$, probability density function (p.d.f.) $f_1(x)$, mean m_1 and Laplace transform (LT) $\widehat{f_1}(s) = \int_0^\infty e^{-sx} f_1(x)\, dx$. Similarly $\{Y_i\}_{i=1}^\infty$ are the positive claim severities from the second class, also assumed i.i.d. r.v., with common d.f. $F_2(x) = \mathbb{P}(Y \leq x)$, p.d.f. $f_2(x)$, mean m_2 and LT $\widehat{f_2}(s) = \int_0^\infty e^{-sx} f_2(x)\, dx$.

The claim number process $\{N_1(t)\}_{t=0}^\infty$ is assumed to be Poisson with parameter λ. More specifically, the corresponding claim inter-arrival times, denoted by $\{W_i\}_{i=1}^\infty$, are i.i.d. exponentially distributed r.v. with parameter λ. In addition, $\{N_2(t)\}_{t=0}^\infty$ is a renewal process with i.i.d. claim inter-arrival times $\{V_i\}_{i=1}^\infty$, which are independent of $\{W_i\}_{i=1}^\infty$ and generalized Erlang(2) distributed r.v., i.e. $V_i = L_{i1} + L_{i2}$, where $\{L_{i1}\}_{i=1}^\infty$ are i.i.d exponentially distributed r.v. with parameter λ_1, while $\{L_{i2}\}_{i=1}^\infty$ are i.i.d. exponentially distributed r.v. with parameter λ_2, (usually $\lambda_1 \neq \lambda_2$).

We finally assume that $\{X_i\}_{i=1}^\infty$ and $\{Y_i\}_{i=1}^\infty$ are mutually independent, independent of the claim number processes $\{N_1(t)\}_{t=0}^\infty$, $\{N_2(t)\}_{t=0}^\infty$.

Let $T = \inf\{t \geq 0 : U(t) < 0\}$ ($T = \infty$ if the set is empty) be the time of ruin, and $\psi(u) = \mathbb{P}(T < \infty | U(0) = u)$, $u \geq 0$, be the ruin probability. The Gerber-Shiu expected discounted penalty function is defined as

$$\phi(u) = \mathbb{E}\left(e^{-\delta T} w(U(T-), |U(T)|) 1_{(T<\infty)} | U(0) = u\right), \quad u \geq 0, \tag{3}$$

where $\delta \geq 0$ is interpreted as the force of interest, $U(T-)$ is the surplus immediately before ruin, $|U(T)|$ is the deficit at ruin, $T-$ is the left limit of T, $w(x, y)$ is a non-negative bivariate function of $x, y \geq 0$ and $1_{(\cdot)}$ represents the indicator function. It is well known that the expected discounted penalty function (3) provides a unified approach to many important quantities related to the ruin time.

In the classical risk theory, due to the memory less property of the exponentially distributed claim inter-arrival, the Gerber-Shiu function is time homogeneous. However in our risk model, due to the generalized Erlang(2) distributional assumption for the inter-arrival claims of the second class, our risk process is no longer time homogeneous. Thus for the expected discount penalty function defined in (3), we assume that a claim from the second class occurs exactly at time 0.

More generally we can define the Gerber-Shiu function ,denoted by $\phi(u, \tau)$, as a bivariate function of the current reserve u and the length of time τ, elapsed since the time of the last claim form the second class (the surplus process renews itself at these points).We are interested in the Gerber-Shiu function at time 0, that is $\phi(u, 0) = \phi(u)$ and the time of realization of L_{i1}, $i = 1, 2$, defined by

$$\phi_1(u) = \mathbb{E}\left(e^{-\delta(T-t)} w\left(U\left(T-\right), \left|U\left(T\right)\right|\right) 1_{(T<\infty)} \left|U(t) = u\right., L_{11} = t\right). \tag{4}$$

Then by the law of total probability we have

$$\phi(u, \tau) = \phi(u)\mathbb{P}(L_{11} > \tau) + \phi_1(u)\mathbb{P}(L_{11} < \tau) = e^{-\lambda_1 \tau}\phi(u) + (1 - e^{-\lambda_1 \tau})\phi_1(u).$$

The rest of the paper is organized as follows: In section 2, we derive an integro-differential equation system for Gerber-Shiu function for the surplus-dependent premium rate case. Using this a piecewise integro-differential equation system for the Gerber-Shiu function is obtained under the multi-layer dividend strategy. In section 3, we show that the solution to the above piecewise integro-differential equation system can be expressed as a particular solution to the non-homogeneous integro-differential equation system plus a linear combination solutions to the corresponding homogeneous integro-differential equation system. Finally, in section 4 we obtain explicit recursive formulas for the Gerber-Shiu functions.

2 Integro-differential equation systems for the Gerber-Shiu functions

In this section, at first we derive an integro-differential equation system for the Gerber-Shiu functions $\phi(u)$ and $\phi_1(u)$ defined in (3) and (4) respectively for the general surplus process defined by (1)-(2), where the premium rate is surplus dependent. Using this, by assuming that the premium rate is constant whenever the surplus is between two consecutive layers, we obtain as a special case the integro-differential equation system that satisfy the Gerber-Shiu functions for the corresponding risk model under the multi-layer dividend strategy.

Theorem 1. *For $u \geq 0$, if $c(u)$ is differentiable at u, then the Gerber-Shiu expected discounted penalty functions $\phi(u)$ and $\phi_1(u)$ satisfy the following integro-differential equation system*

$$\begin{aligned}
c(u)\phi'(u) &= -\lambda_1\phi_1(u) + (\lambda + \lambda_1 + \delta)\phi(u) - \lambda \int_0^u \phi(u-x) f_1(x) dx - \lambda w_1(u), \\
c(u){\phi_1}'(u) &= -\lambda_2 \int_0^u \phi(u-x) f_2(x) dx - \lambda_2 w_2(u) + (\lambda + \lambda_2 + \delta)\phi_1(u) \\
&\quad - \lambda \int_0^u \phi_1(u-x) f_1(x) dx - \lambda w_1(u),
\end{aligned} \tag{5}$$

where

$$w_j(x) = \int_x^\infty w(x, y-x) f_j(y) dy = \int_0^\infty w(x, y) f_j(x+y) dy, \quad j = 1, 2. \tag{6}$$

Proof. Let $M = W_1 \wedge L_{11}$ and conditioning on the events $\{M = t, M = L_{11}\}$ and $\{M = t, M = W_1\}$, one has for $u \geq 0$

$$\begin{aligned}
\phi(u) &= \int_0^\infty e^{-\delta t} \mathbb{P}(M = t, M = L_{11}) \phi_1(U(t)) dt \\
&\quad + \int_0^\infty e^{-\delta t} \mathbb{P}(M = t, M = W_1) \Big(\int_0^{U(t)} \phi(U(t) - x) f_1(x) dx \\
&\quad + \int_{U(t)}^\infty w(U(t), x - U(t)) f_1(x) dx \Big) dt.
\end{aligned}$$

Since, $\mathbb{P}(M = W_1) = \mathbb{P}(W_1 < L_{11}) = \frac{\lambda}{\lambda+\lambda_1}$, $\mathbb{P}(M = L_{11}) = \mathbb{P}(W_1 > L_{11}) = \frac{\lambda_1}{\lambda+\lambda_1}$ and $\mathbb{P}(M > t | M = W_1) = \mathbb{P}(M > t | M = L_{11}) = e^{-(\lambda+\lambda_1)t}$, it yields that

$$\phi(u) = \int_0^\infty \lambda_1 e^{-(\lambda+\lambda_1+\delta)t} \phi_1(U(t)) dt + \int_0^\infty \lambda e^{-(\lambda+\lambda_1+\delta)t} \xi_1(U(t)) dt, \tag{7}$$

where $\xi_j(u) = \int_0^u \phi(u-x) f_j(x) dx + w_j(u), \ j = 1, 2$. Now, using the fact that $U(t)$ is an increasing function and differentiable except at a finite number of points, we may change variable $s = U(t)$ to Eq. (7). Noting that $dU(t) = c(U(t))dt$ and $U(0) = u$ imply that $ds = c(s)dt$ and that $t = \int_u^s [c(x)]^{-1} dx$ and hence

$$\begin{aligned}
\phi(u) &= \lambda_1 \int_u^\infty e^{-(\lambda+\lambda_1+\delta) \int_u^s [c(x)]^{-1} dx} \phi_1(s) [c(s)]^{-1} ds \\
&\quad + \lambda \int_u^\infty e^{-(\lambda+\lambda_1+\delta) \int_u^s [c(x)]^{-1} dx} \xi_1(s) [c(s)]^{-1} ds.
\end{aligned}$$

Differentiating the above equation w.r.t. u one finds the integro-differential equation (5) for $\phi(u)$.

Similarly let $Z = W_1 \wedge L_{12}$. Then conditioning on the events $\{Z = t, Z = L_{12}\}$ and $\{Z = t, Z = W_1\}$, we obtain for $u \geq 0$

$$\begin{aligned}
\phi_1(u) &= \int_0^\infty e^{-\delta t} \mathbb{P}(Z = t, Z = L_{12}) \Big(\int_0^{U(t)} \phi(u-x) f_2(x) dx \\
&\quad + \int_{U(t)}^\infty w(U(t), x - U(t)) f_2(x) dx \Big) \\
&\quad + \int_0^\infty e^{-\delta t} \mathbb{P}(Z = t, Z = W_1) \Big(\int_0^{U(t)} \phi_1(u-x) f_1(x) dx \\
&\quad + \int_{U(t)}^\infty w(U(t), x - U(t)) f_1(x) dx \Big) \\
&= \lambda_2 \int_0^\infty e^{-(\lambda+\lambda_2+\delta)t} \xi_2(U(t)) dt + \lambda \int_0^\infty e^{-(\lambda+\lambda_2+\delta)t} \mu(U(t)) dt,
\end{aligned} \tag{8}$$

where $\mu(u) = \int_0^u \phi_1(u-x) f_1(x) dx + w_1(u)$. Then by changing variable $s = U(t)$ to Eq. (8) we have that

$$\phi_1(u) = \lambda_2 \int_u^\infty e^{-(\lambda+\lambda_2+\delta) \int_u^s [c(x)]^{-1} dx} \xi_2(s) [c(s)]^{-1} ds + \lambda \int_u^\infty e^{-(\lambda+\lambda_2+\delta) \int_u^s [c(x)]^{-1} dx} \mu(s) [c(s)]^{-1} ds.$$

Differentiating w.r.t. u we get immediately Eq. (5) for $\phi_1(u)$. □

Remark 1. *Note that the two classes risk model given by Eq.* (1)-(2)*, contains as special cases both the compound Poisson and the renewal generalized Erlang*$(2, \lambda_1, \lambda_2)$ *risk model.*

(i) For $\lambda_1, \lambda_2 \to 0$ *it is straightforward to check that Eq.* (5)*, lead to Eq. (3.2) of Lin and Sendova (2007) which is the integro-differential equation for the Gerber-Shiu function in the compound Poisson risk model under the multi-layer dividend strategy.*

(ii) Taking $\lambda \to 0$ *into the second equation in (5), and then substituting into the first part of* (5) *and differentiating the resulting equation w.r.t.* u *we get the integro-differential equation (2.1) of Theorem 1 of Yang and Zhang (2007) for the renewal generalized Erlang(2) risk model under the multi-layer dividend strategy.*

Now, similar to Albrecher and Hantinger (2007) (see also Lin and Sendova (2007), Yang and Zhang (2007)), we consider the multi-layer risk model by defining n-layers, say $0 = b_0 < b_1 < ... < b_{n-1} < b_n = \infty$. We assume that the premium is collected with rate c_i whenever the surplus is in the layer i, i.e., between the thresholds b_{i-1} and b_i, $i = 1, \ldots, n$. To separate this special case from the general surplus dependent risk model, we define the surplus process $U_{\mathbf{b}}(t)$, given by

$$dU_{\mathbf{b}}(t) = c_i dt - dS(t), \quad b_{i-1} \le U_{\mathbf{b}}(t) < b_i, \qquad i = 1, \ldots, n, \tag{9}$$

where the aggregate process $S(t)$ is given by Eq. (2). Also, we assume that for $i = 1, \ldots, n$, $c_i > \lambda m_1 + [\lambda_1\lambda_2/(\lambda_1+\lambda_2)]m_2$ meaning that there is a positive security loading θ_i, such that $1/(1+\theta_i) = \big[\lambda m_1 + \lambda_1\lambda_2 m_2/(\lambda_1 + \lambda_2)\big]/c_i$. Under this modification, let $T_{\mathbf{b}} = \inf\{t \ge 0 : U_{\mathbf{b}}(t) < 0\}$ to be the time of ruin and

$$\phi(u;\mathbf{b}) = \left(e^{-\delta T_{\mathbf{b}}} w(U_{\mathbf{b}}(T_{\mathbf{b}}-, |U(T_{\mathbf{b}})|) 1_{(T_{\mathbf{b}}<\infty)} \big| U_{\mathbf{b}}(0) = u\right),$$

to be the expected discounted penalty function for the multi-layer risk model (9). Also, as in the previous section we define the expected discounted penalty function in the time of realization of L_{i1}, $i = 1, 2$, given by

$$\phi_1(u;\mathbf{b}) = \left(e^{-\delta (T_{\mathbf{b}}-t)} w(U_{\mathbf{b}}(T_{\mathbf{b}}-, |U(T_{\mathbf{b}})|) 1_{(T_{\mathbf{b}}<\infty)} \big| U_{\mathbf{b}}(t) = u, L_{11} = t\right).$$

Theorem 2. *For* $b_{i-1} < u < b_i$, $i = 1, 2, ..., n$*, the Gerber-Shiu expected discounted penalty functions* $\phi(u, \mathbf{b})$ *and* $\phi_1(u, \mathbf{b})$ *satisfy the following piecewise linear integro-differential equation system*

$$\begin{aligned} c_i\phi'(u;\mathbf{b}) &= -\lambda_1\phi_1(u;\mathbf{b}) + (\lambda+\lambda_1+\delta)\phi(u;\mathbf{b}) - \lambda\int_0^u \phi(u-x,\mathbf{b}) f_1(x)dx - \lambda w_1(u), \\ c_i\phi_1'(u;\mathbf{b}) &= -\lambda_2\int_0^u \phi(u-x;\mathbf{b}) f_2(x)dx - \lambda_2 w_2(u) + (\lambda+\lambda_2+\delta)\phi_1(u;\mathbf{b}) \\ &\quad - \lambda\int_0^u \phi_1(u-x;\mathbf{b}) f_1(x)dx - \lambda w_1(u), \end{aligned} \tag{10}$$

with boundary conditions

$$c_{i+1}\lim_{u\to b_i+}\phi'(u;\mathbf{b}) = c_i\lim_{u\to b_i-}\phi'(u;\mathbf{b}), \quad c_{i+1}\lim_{u\to b_i+}\phi_1'(u;\mathbf{b}) = c_i\lim_{u\to b_i-}\phi_1'(u;\mathbf{b}), \tag{11}$$

where $w_j(u)$ *for* $j = 1, 2$ *are given by Eq.* (6)

Proof. Substituting the assumption $c(u) = c_i$ whenever $b_{i-1} \le U_{\mathbf{b}}(t) < b_i$ into Theorem 1, the integro-differential equation system (10) is straightforward. To obtain the boundary conditions (11), first note that from (7), (8), $\phi(u;\mathbf{b})$, $\phi_1(u;\mathbf{b})$ are always continuous in u, even at the seeming discontinuity points. Thus under the multi-layer dividend strategy, the Gerber-Shiu functions $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ are continuous at each b_i, $i = 1, \ldots, n$. Furthermore $\phi(u;\mathbf{b})$, $\phi_1(u;\mathbf{b})$ are not differentiable at b_i's but their left and right derivatives exist. Then, from the integro-differential equation system (10) and the continuity of $\phi(u,\mathbf{b})$ and $\phi_1(u,\mathbf{b})$ at b_i's, it is not difficult to see that the boundary conditions given by (11) hold. □

The existence of the right derivative of $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ allows to define the system (10) for all u, in which the derivatives $\phi'(u;\mathbf{b})$ and $\phi_1'(u;\mathbf{b})$ w.r.t. u are defined as the right derivatives and the constraint $b_{i-1} < u < b_i$ is replaced by $b_{i-1} \le u < b_i$ for each $i = 1, \ldots, n$. Therefore in (10) the derivatives of the derivatives of $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ at $u = b_i$, $i = 1, \ldots, n-1$ are assumed to be right derivatives.

3 Analysis of the piece-wise integro-differential equation system

The main goal of this section is to analyze the piecewise integro-differential equation sysytem (10). We first introduce some auxiliary results.

Lemma 1. *Let δ be strictly positive and let*

$$\ell_i(s) = \lambda_1\lambda_2\widehat{f}_2(s), \tag{12}$$

where $\ell_i(s) = \left[c_i s + \lambda\widehat{f}_1(s) - (\lambda+\lambda_1+\delta)\right]\left[c_i s + \lambda\widehat{f}_1(s) - (\lambda+\lambda_2+\delta)\right]$. For each $i = 1, 2, ..., n$, Eq. (12) has exactly two distinct positive real roots, say $r_{1i}(\delta)$ and $r_{2i}(\delta)$, which are located in the right half of the complex plane.

Proof. See proof of Theorem 1 of Li and Lu (2005). □

If we denote the root with the smallest real part by $r_{1i}(\delta)$, then $\lim_{\delta\to 0} r_{1i}(\delta) = 0$. In the rest of the paper we assume that the two roots are distinct, and for simplicity we denote them as $r_{ji}(\delta) \equiv r_{ji}$, $j = 1, 2$, $i = 1, \ldots, n$.

Also, to analyze the piecewise integro-differential equation system (10) we introduce a commonly used operator in risk theory, namely the Dickson-Hipp operator T_s. Let f be a real-valued integrable function and s be a nonnegative real number (or a complex with nonnegative real part). Then, the operator $T_s f$ is defined as

$$T_s f(x) = \int_x^\infty e^{-s(y-x)} f(y)dy, \quad x \ge 0.$$

For more details about T_s operators, we refer to Dickson and Hipp (2001), Li and Garrido (2004a) and the references therein. Also, for notation reasons, we define the following operators for $i = 1, 2, \ldots, n$

$$\begin{aligned}
\mathcal{S}_i &= \textstyle\prod_{j=1}^2 T_{r_{ji}} = \frac{T_{r_{2i}} - T_{r_{1i}}}{r_{1i} - r_{2i}},\\
\mathcal{M}_{1,i} &= \lambda c_i T_{r_{1i}} + \lambda(\lambda+\lambda_2+\delta - c_i r_{2i} - \lambda\widehat{f}_1(r_{1i}))\mathcal{S}_i,\\
\mathcal{M}_{2,i} &= \frac{\lambda}{\lambda_1}(c_i r_{2i} + \lambda\widehat{f}_1(r_{2i}) - \lambda - \lambda_1 - \lambda_2 - \delta)\mathcal{S}_i.
\end{aligned}$$

Hereafter, we constrain the Gerber-Shiu functions $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ to the class such that $T_{r_{ki}} w_j(u) < \infty$, (and consequently $\mathcal{S}_i w_j(u) < \infty$), $i = 1, \ldots, n$, $k, j = 1, 2$. Similar to the method of Lin and Sendova (2007) , see also in Yang and Zhang (2007), we relax the constraints $b_{i-1} \le u < b_i$ to $b_{i-1} \le u$ in the non-homogeneous integro-differential equation system in (10), and we let $\Phi_i(u)$, $\Phi_{1,i}(u)$ (with $T_{r_j}\Phi_i(b_{i-1}) < \infty$, $T_{r_{ji}}\Phi_{1,i}(b_{i-1}) < \infty$) be the solution of the following non-homogeneous integro-differential equation system for $i = 1, \ldots, n$

$$
\begin{aligned}
c_i\Phi_i{}'(u) &= -\lambda_1\Phi_{1,i}(u) + (\lambda+\lambda_1+\delta)\Phi_i(u) - \lambda\int_0^{u-b_{i-1}}\Phi_i(u-x)f_1(x)dx \\
&\quad - \lambda\int_{u-b_{i-1}}^{u}\phi(u-x)f_1(x)dx - \lambda w_1(u), \quad u \geq b_{i-1} \\
c_i\Phi_{1,i}'(u) &= -\lambda_2\int_0^{u-b_{i-1}}\Phi_i(u-x)f_2(x)dx - \lambda_2\int_{u-b_{i-1}}^{u}\phi(u-x)f_2(x)dx - \lambda_2 w_2(u) \\
&\quad + (\lambda+\lambda_2+\delta)\Phi_{1,i}(u) - \lambda w_1(u) - \lambda\int_0^{u-b_{i-1}}\Phi_{1,i}(u-x)f_1(x)dx \\
&\quad - \lambda\int_{u-b_{i-1}}^{u}\phi_1(u-x)f_1(x)dx, \quad u \geq b_{i-1}.
\end{aligned} \tag{13}
$$

Then, the solution to the non homogeneous piecewise integro-differential equation system (10), with boundary conditions (11), heavily depends on the solutions $\Phi_i(u)$, $\Phi_{1,i}(u)$ as well as on the solution of the following associated homogeneous integro-differential equation system for $i = 1, \ldots, n$

$$
\begin{aligned}
&c_i\upsilon_i{}'(u) - (\lambda+\lambda_1+\delta)\upsilon_i(u) + \lambda\int_0^{u-b_{i-1}}\upsilon_i(u-x)f_1(x)dx + \lambda_1\upsilon_{1,i}(u) = 0, \\
&c_i\upsilon_{1,i}'(u) + \lambda_2\int_0^{u-b_{i-1}}\upsilon_i(u-x)f_2(x)dx + \lambda\int_0^{u-b_{i-1}}\upsilon_{1,i}(u-x)f_1(x)dx \\
&\qquad\qquad - (\lambda+\lambda_2+\delta)\upsilon_{1,i}(u) = 0,
\end{aligned} \tag{14}
$$

It follows from the general theory of the differential equations the solution of the homogeneous system (14) is of the form

$$
\begin{pmatrix} \upsilon_i(u) \\ \upsilon_{1,i}(u) \end{pmatrix} = k_{i,1}\begin{pmatrix} \upsilon_{i,11}(u) \\ \upsilon_{i,21}(u) \end{pmatrix} + k_{i,2}\begin{pmatrix} \upsilon_{i,12}(u) \\ \upsilon_{i,22}(u) \end{pmatrix}, \quad u \geq b_{i-1}, \tag{15}
$$

where $[\upsilon_{i,11}(u), \upsilon_{i,21}(u)]'$ and $[\upsilon_{i,12}(u), \upsilon_{i,22}(u)]'$ are independent solutions of the homogeneous system (14) and $k_{i,1}$, $k_{i,2}$ some arbitrary constants. Let $\upsilon_{i,kl}(b_{i-1}) = 1_{(i=j)}$, $i = 1, \ldots, n$, $k, l \in \{1, 2\}$. Then, it is easy to see that $[\upsilon_{i,11}(u), \upsilon_{i,21}(u)]'$ and $[\upsilon_{i,12}(u), \upsilon_{i,22}(u)]'$ are two linearly independent solutions, and thus the solution to the non homogeneous system (10) can be expressed as, for $i = 1, \ldots, n$,

$$
\begin{aligned}
\phi(u;\mathbf{b}) &= \Phi_i(u) + \eta_{i,1}\upsilon_{i,11}(u) + \eta_{i,2}\upsilon_{i,12}(u), \quad b_{i-1} \leq u \leq b_i, \\
\phi_1(u;\mathbf{b}) &= \Phi_{1,i}(u) + \eta_{i,1}\upsilon_{i,21}(u) + \eta_{i,2}(b)\upsilon_{i,22}(u), \quad b_{i-1} \leq u \leq b_i,
\end{aligned} \tag{16}
$$

where $\Phi_i(u)$, $\Phi_{1,i}(u)$ are some particular solutions of the non-homogeneous system (13), and $\eta_{i,1}$, $\eta_{i,2}$ are constant coefficients determined by the boundary conditions (11). We remark that $\Phi_1(u)$, $\Phi_{1,1}(u)$ are the Gerber-Shiu expected discounted penalty functions in the absence of dividend barriers, which have been studied in Li and Lu (2005) for exponential claim amount severities and in Chadjiconstantinidis and Papaioannou (2009) for claim severities having rational Laplace Transforms (LT).

3.1 Defective renewal equations Φ_i(u), $\Phi_{1,i}$(u) and their solutions

In this section, we shall prove (although the process $S(t)$ is neither a compound renewal nor a compound Poisson one) that $\Phi_i(u)$ and $\Phi_{1,i}$(u) satisfy some defective renewal equations. Then their exact representations can be easily obtained according to Theorem 2.1 of Lin and Willmot (1999).

Hereafter, a change of variable $y = u - b_{i-1}$ and $Q_i(y) = Q_i(u - b_{i-1}) = \Phi_i(u)$, $Q_{1,i}(y) = Q_{1,i}(u - b_{i-1}) = \Phi_{1,i}(u)$ brings integro-differential equation system (13) , for $y \geq 0$, $i = 1, \ldots, n$, into

$$
\begin{aligned}
c_i Q_i'(y) &= -\lambda_1 Q_{1,i}(y) + (\lambda + \lambda_1 + \delta) Q_i(y) - \lambda \int_0^y Q_i(y-x) f_1(x)dx - A_i(y), \\
c_i Q_{1,i}'(y) &= -\lambda_2 \int_0^y Q_i(y-x) f_2(x)dx + (\lambda + \lambda_2 + \delta) Q_{1,i}(y) \\
&\quad - \lambda \int_0^y Q_{1,i}(y-x) f_1(x)dx - A_{1,i}(y),
\end{aligned} \tag{17}
$$

where $A_i(y) = \lambda \int_0^{b_{i-1}} \phi(x) f_1(y + b_{i-1} - x)dx + \lambda w_1(y + b_{i-1})$ and $A_{1,i}(y) = \lambda_2 \int_0^{b_{i-1}} \phi(x) f_2(y + b_{i-1} - x)dx + \lambda_2 w_2(y + b_{i-1}) + \lambda \int_0^{b_{i-1}} \phi_1(x) f_1(y + b_{i-1} - x)dx + \lambda w_1(y + b_{i-1})$.

Further, for $\Re(s) \geq 0$, let $\widehat{Q}_i(s) = \int_0^\infty e^{-sy} Q_i(y)dy$, $\widehat{Q}_{1,i}(s) = \int_0^\infty e^{-sy} Q_{1,i}(y)dy$, $\widehat{A}_i(s) = \int_0^\infty e^{-sy} A_i(y)dy$, and $\widehat{A}_{1,i}(s) = \int_0^\infty e^{-sy} A_{1,i}(y)dy$ to be the LT of $Q_i(y)$, $Q_{1,i}(y)$, $A_i(y)$, and $A_{1,i}(y)$ respectively. Also, by the form of $A_i(y)$ and $A_{1,i}(y)$ it is not difficult to see that

$$
\begin{aligned}
\widehat{A}_i(s) &= \lambda \int_0^{b_{i-1}} \phi(x) T_s f_1(b_{i-1} - x)dx + \lambda T_s w_1(b_{i-1}), \\
\widehat{A}_{1,i}(s) &= \lambda_2 \int_0^{b_{i-1}} \phi(x) T_s f_2(b_{i-1} - x)dx + \lambda_2 T_s w_2(b_{i-1}) \\
&\quad + \lambda \int_0^{b_{i-1}} \phi_1(x) T_s f_1(b_{i-1} - x)dx + \lambda T_s w_1(b_{i-1}).
\end{aligned}
$$

Then, by taking the LT in both sides of equations in (17), and by solving the resulting system yields for $i = 1, \ldots, n$

$$
\begin{aligned}
\widehat{Q}_i(s) &= \frac{[c_i s - (\lambda + \lambda_2 + \delta) + \lambda \widehat{f}_1(s)][c_i Q_i(0) - \widehat{A}_i(s)] - \lambda_1 [c_i Q_{1,i}(0) - \widehat{A}_{1,i}(s)]}{\ell_i(s) - \lambda_1 \lambda_2 \widehat{f}_2(s)}, \\
\widehat{Q}_{1,i}(s) &= \frac{[c_i s - (\lambda + \lambda_1 + \delta) + \lambda \widehat{f}_1(s)][c_i Q_{1,i}(0) - \widehat{A}_{1,i}(s)] - \lambda_2 \widehat{f}_2(s)[c_i Q_i(0) - \widehat{A}_i(s)]}{\ell_i(s) - \lambda_1 \lambda_2 \widehat{f}_2(s)}.
\end{aligned} \tag{18}
$$

For the complete solution of (18), we need to determine the quantities $Q_i(0)$, $Q_{1,i}(0)$. By the fact that $T_{r_{ji}} \Phi_i(b_{i-1}) < \infty$, $T_{r_{ji}} \Phi_{1,i}(b_{i-1}) < \infty$, implies that $\widehat{Q}_i(r_{ji}) < \infty$, $\widehat{Q}_{1,i}(r_{ji}) < \infty$, $j = 1, 2$. Therefore, for all $\Re(s) \geq 0$ the numerator in (18) is zero for $s = r_{1i}$ and r_{2i}, i.e.

$$
[c_i r_{ji} - (\lambda + \lambda_2 + \delta) + \lambda \widehat{f}_1(r_{ji})][c_i Q_i(0) - \widehat{A}_i(r_{ji})] - \lambda_1 [c_i Q_{1,i}(0) - \widehat{A}_{1,i}(r_{ji})] = 0, \quad j = 1, 2. \tag{19}
$$

By solving this linear equation system for $Q_i(0)$, $Q_{1,i}(0)$, we get for $i = 1, \ldots, n$

$$
\begin{aligned}
Q_i(0) &= \frac{T_{r_{2i}} A_i(0)}{c_i} + \frac{\lambda_1 \mathcal{S}_i A_{1,i}(0) - \left[c_i r_{1i} + (\lambda + \lambda_2 + \delta) + \lambda \widehat{f}_1(r_{1i})\right] \mathcal{S}_i A_i(0)}{c_i (c_i - \lambda \mathcal{S}_i f_1(0))}, \\
Q_{1,i}(0) &= \frac{\left[c_i r_{2i} - (\lambda + \lambda_2 + \delta) + \lambda \widehat{f}_1(r_{2i})\right] \left[c_i Q_i(0) - \widehat{A}_i(r_{2i})\right] + \lambda_1 \widehat{A}_{1,i}(r_{2i})}{\lambda_1 c_i}.
\end{aligned} \tag{20}
$$

Now it can be easily seen that for where for $\ell = \begin{cases} 1,2 & ,k=1 \\ 2 & ,k=2 \end{cases}$, it holds

$$
\prod_{j=k}^{\ell} T_{r_{ji}} \widehat{A}_i(s) = \lambda \int_0^{b_{i-1}} \phi(x) T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}} \right) f_1(b_{i-1} - x)dx + \lambda T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}} \right) w_1(b_{i-1}),
$$

and

$$\begin{aligned}
\prod_{j=k}^{\ell} T_{r_{ji}} \widehat{A}_{1,i}(s) &= \lambda_2 \int_0^{b_{i-1}} \phi(x) T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}}\right) f_2(b_{i-1}-x)dx + \lambda_2 T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}}\right) w_2(b_{i-1}) \\
&\quad + \lambda \int_0^{b_{i-1}} \phi_1(x) T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}}\right) f_1(b_{i-1}-x)dx + \lambda T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}}\right) w_1(b_{i-1}).
\end{aligned}$$

Proposition 1. *For $\Re(s) > 0$ the LT of $\widehat{Q}_i(s)$ and $\widehat{Q}_{1,i}(s)$, given in (20), can be expressed as*

$$\widehat{Q}_i(s) = \frac{\widehat{G}_i(s)}{{c_i}^2 - \widehat{n}_i(s)}, \qquad \widehat{Q}_{1,i}(s) = \widehat{Q}_i(s) + \frac{\widehat{G}_{1,i}(s)}{{c_i}^2 - \widehat{n}_i(s)}, \tag{21}$$

where

$$\begin{aligned}
\widehat{n}_i(s) &= 2\lambda c_i T_{r_{1i}} \widehat{f}_1(s) + \lambda \left((\lambda_1 + \lambda_2 + 2\lambda + 2\delta) - 2\lambda c_i r_{2i} - \lambda \widehat{f}_1(r_{1i}) - \lambda \widehat{f}_1(r_{2i})\right) \mathcal{S}_i \widehat{f}_1(s) \\
&\quad - \lambda^2 T_{r_{1i}} \widehat{f}_1(s) T_{r_{2i}} \widehat{f}_1(s) - \lambda_1 \lambda_2 \mathcal{S}_i \widehat{f}_2(s), \\
\widehat{G}_i(s) &= c_i T_{r_{1i}} \widehat{A}_i(s) + \left(\lambda + \lambda_2 + \delta - c_i r_{2i} - \lambda \widehat{f}_1(r_{1i})\right) \mathcal{S}_i \widehat{A}_i(s) + \lambda_1 \mathcal{S}_i \widehat{A}_{1,i}(s) + \lambda \big[c_i Q_i(0) \\
&\quad - \widehat{A}_i(r_{2i})\big] \mathcal{S}_i \widehat{f}_1(s) - \lambda T_{r_{1i}} \widehat{f}_1(s) T_{r_{2i}} \widehat{A}_i(s),
\end{aligned}$$

and

$$\begin{aligned}
\widehat{G}_{1,i}(s) &= c_i \left(T_{r_{1i}} \widehat{A}_{1,i}(s) - T_{r_{1i}} \widehat{A}_{1,i}(s)\right) - \lambda_2 \left(1 - \widehat{f}_2(r_{1i})\right) \mathcal{S}_i \widehat{A}_i(s) - \lambda T_{r_{1i}} \widehat{f}_1(s) \Big(T_{r_{2i}} \widehat{A}_{1,i}(s) \\
&\quad - T_{r_{2i}} \widehat{A}_{1,i}(s)\Big) + \lambda_2 T_{r_{1i}} \widehat{f}_2(s) T_{r_{2i}} \widehat{A}_i(s) + \left(c_i Q_i(0) - \widehat{A}_i(r_{2i})\right) \Bigg(\frac{\lambda}{\lambda_1}\big(r_{2i} + \lambda \widehat{f}_1(r_{2i}) \\
&\quad - (\lambda + \lambda_1 + \lambda_2 + \delta)\big) \mathcal{S}_i \widehat{f}_1(s) - \lambda_2 \mathcal{S}_i \widehat{f}_2(s)\Bigg) + \left(\lambda + \delta - c_i r_{2i} - \lambda \widehat{f}_1(r_{1i})\right) \Big(\mathcal{S}_i \widehat{A}_{1,i}(s) \\
&\quad - \mathcal{S}_i \widehat{A}_{1,i}(s)\Big).
\end{aligned}$$

Proof. Since the equations given in (18) have the same form with Eqs. (11)-(12) of Chadjiconstantinidis and Papaioannou (2009), from Proposition 1 of their paper we recover the result. □

Now, in order to derive the defective renewal equations for $\Phi_i(u)$ and $\Phi_{1,i}(u)$, we need to invert the LT $\widehat{Q}_i(s)$, $\widehat{Q}_{1,i}(s)$ and $\widehat{n}_i(s)$ of Proposition 1 w.r.t. s. To do this, we need first to invert the LT $\widehat{G}_i(s)$, $\widehat{G}_{1,i}(s)$ and $\widehat{n}_i(s)$.

Note that for an integrable function $f(x)$ it is not difficult to check that for $\Re(s) \geq 0$, for the same values of ℓ as previously

$$T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}} f\right)(b_{i-1}) = \int_0^{\infty} e^{-sy} \left(\prod_{j=k}^{\ell} T_{r_{ji}} f\right)(y + b_{i-1}) dy,$$

and similarly for two integrable functions $f_1(x)$ and $f_2(x)$, it holds that

$$\int_0^{b_{i-1}} f_1(x) T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}} f_2\right)(b_{i-1}) dx = \int_0^{\infty} e^{-sy} \int_0^{b_{i-1}} f_1(x) \left(\prod_{j=k}^{\ell} T_{r_{ji}} f_2\right)(y + b_{i-1} - x) dx dy.$$

Hence it is easy to see that,

$$\mathcal{L}^{-1}\left(T_s \left(\prod_{j=k}^{\ell} T_{r_{ji}} f\right)(b_{i-1})\right) = \prod_{j=k}^{\ell} T_{r_{ji}} f(y + b_{i-1}), \tag{22}$$

and

$$\mathcal{L}^{-1}\left(\int_0^{b_{i-1}} f_1(x) T_s\left(\prod_{j=k}^{\ell} T_{r_{ji}} f_2\right)(b_{i-1}) dx\right) = \int_0^{b_{i-1}} f_1(x)\left(\prod_{j=k}^{\ell} T_{r_{ji}} f_2\right)(y+b_{i-1}-x) dx. \quad (23)$$

Thus, using Eqs. (22) - (23), the inversion of $\widehat{G}_i(s)$, after collecting the terms involving $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$, turns out to be

$$\frac{G_i(y)}{{c_i}^2} = \int_0^{b_{i-1}} \phi(x;\mathbf{b}) Z_{1,i}(y+b_{i-1},x) dx + \int_0^{b_{i-1}} \phi_1(x;\mathbf{b}) Z_{2,i}(y+b_{i-1},x) dx + Z_i(y+b_{i-1}), \quad (24)$$

where

$$\begin{aligned} Z_{k,i}(u,x) &= z_{k,i}(u,x) + \frac{\lambda z_{k,i}(b_{i-1})}{c-\lambda \mathcal{S}_i f_1(0)} \mathcal{S}_i f_1(u-b_{i-1}), \qquad k=1,2, \\ Z_i(u,x) &= z_i(u,x) + \frac{\lambda z_i(b_{i-1})}{c-\lambda \mathcal{S}_i f_1(0)} \mathcal{S}_i f_1(u-b_{i-1}), \end{aligned}$$

with

$$\begin{aligned} z_{1,i}(u,x) &= \frac{1}{{c_i}^2}\Bigg(\mathcal{M}_{1,i} f_1(u-x) + \lambda_1\lambda_2 \mathcal{S}_i f_2(u-x) - \lambda^2 T_{r_{2i}} f_1(b_{i-1}-x) \mathcal{S}_i f_1(u-b_{i-1}) \\ &\quad -\lambda^2 \int_0^{u-b_{i-1}} T_{r_{2i}} f_1(u-x-z) T_{r_{1i}} f_1(z) dz\Bigg), \\ z_{2,i}(u,x) &= \frac{\lambda\lambda_1}{{c_i}^2} \mathcal{S}_i f_1(u-x), \\ z_i(u) &= \frac{1}{{c_i}^2}\Bigg(\mathcal{M}_{1,i} w_1(u) + \lambda_1 \mathcal{S}_i\big(\lambda_2 w_2(u) + \lambda w_1(u)\big) - \lambda^2 T_{r_{2i}} w_1(b_{i-1}) \mathcal{S}_i f_1(u-b_{i-1}) \\ &\quad -\lambda^2 \int_0^{u-b_{i-1}} T_{r_{2i}} w_1(u-z) T_{r_{1i}} f_1(z) dz\Bigg). \end{aligned}$$

In a similar way the inversion of $\widehat{G}_{1,i}(s)$, after collecting the terms involving $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$, is given by

$$\frac{G_{1,i}(y)}{{c_i}^2} = \int_0^{b_{i-1}} \phi(x;\mathbf{b}) H_{1,i}(y+b_{i-1},x) dx + \int_0^{b_{i-1}} \phi_1(x;\mathbf{b}) H_{2,i}(y+b_{i-1},x) dx + H_i(y+b_{i-1}), \quad (25)$$

where

$$\begin{aligned} H_i(u,x) &= h_{k,i}(u,x) + \frac{\mathcal{M}_{2,i} f_1(u-b_{i-1}) - \lambda_2 \mathcal{S}_i f_2(u-b_{i-1})}{c-\lambda \mathcal{S}_i f_1(0)} h_{k,i}(b_{i-1}), \qquad k=1,2, \\ H_i(u) &= h_i(u) + \frac{\mathcal{M}_{2,i} f_1(u-b_{i-1}) - \lambda_2 \mathcal{S}_i f_2(u-b_{i-1})}{c-\lambda \mathcal{S}_i f_1(0)} h_i(b_{i-1}), \end{aligned}$$

with

$$\begin{aligned} h_{1,i}(u,x) &= \frac{1}{{c_i}^2}\Bigg(\lambda_2\big(\frac{1}{\lambda}\mathcal{M}_{1,i} - \lambda_2 \mathcal{S}_i\big) f_2(u-x) + \big(\lambda_2 \lambda \widehat{f}_2(r_{1i}) \mathcal{S}_i - \mathcal{M}_{1,i}\big) f_1(u-x) \\ &\quad + \int_0^{u-b_{i-1}} \big[T_{r_{2i}} f_1(u-x-z)\left(\lambda_2 \lambda T_{r_{1i}} f_2(z) + \lambda^2 T_{r_{1i}} f_1(z)\right) \\ &\quad -\lambda\lambda_2 T_{r_{2i}} f_2(u-x-z) T_{r_{1i}} f_1(z)\big] dz - \lambda\big(\mathcal{M}_{2,i} f_1(u-b_{i-1}) - \lambda_2 \mathcal{S}_i f_2(u-b_{i-1})\big) \\ &\quad \times T_{r_{2i}} f_1(b_{i-1}-x)\Bigg), \end{aligned}$$

$$
\begin{aligned}
h_{2,i}(u,x) &= \frac{1}{{c_i}^2}\Bigg(\big(\mathcal{M}_{1,i}-\lambda\lambda_2\mathcal{S}_i\big)f_1(u-x)-\lambda^2\int_0^{u-b_{i-1}} T_{r_{2i}}f_1(u-x-z)T_{r_{1i}}f_1(z)dz\Bigg),\\
h_i(u) &= \frac{1}{{c_i}^2}\Bigg(\big(\frac{1}{\lambda}\mathcal{M}_{1,i}-\lambda_2\mathcal{S}_i\big)w_2(u)-\lambda\lambda_2\big(1-\widehat{f_2}(r_{1i})\big)\mathcal{S}_i w_1-\lambda\big(\mathcal{M}_{2,i}f_1(u-b_{i-1})\\
&\quad -\lambda_2\mathcal{S}_i f_2(u-b_{i-1})\big)T_{r_{2i}}w_2(b_{i-1})+\lambda\lambda_2\int_0^{u-b_{i-1}}\sum_{\nu=1,j\neq\nu}^{2}(-1)^{\nu}T_{r_{2i}}w_j(u-z)T_{r_{1i}}f_{\nu}(z)dz\Bigg).
\end{aligned}
$$

Now, inverting both equations in (21) w.r.t. s lead us to the following renewal equations

$$
\begin{aligned}
Q_i(y) &= \int_0^y Q_i(y-x)\frac{n_i(x)}{{c_i}^2}dx+\frac{G_i(y)}{{c_i}^2}, &(26)\\
Q_{1,i}(y) &= \int_0^y Q_{1,i}(y-x)\frac{n_i(x)}{{c_i}^2}dx+\frac{G_i(y)+G_{1,i}(y)}{{c_i}^2}, &(27)
\end{aligned}
$$

where $G_i(y)$ and $G_{1,i}(y)$ are given by Eqs. (24) and (25) respectively,

$$
\begin{aligned}
n_i(u) &= 2\lambda c_i T_{r_{1i}}f_1(u)+\lambda\Big(\lambda_1+\lambda_2+2\lambda+2\delta-2\lambda c_i r_{2i}-\lambda\widehat{f_1}(r_{1i})-\lambda f_1(r_{2i})\Big)\mathcal{S}_i f_1(u)\\
&\quad -\lambda^2(T_{r_{1i}}f_1\star T_{r_{2i}}f_1)(u)+\lambda_1\lambda_2\mathcal{S}_i f_2(u),
\end{aligned}
$$

and $\star$ denotes the convolution operator for real functions.

A change of variable $y=u-b_{i-1}$, $Q_i(y)=Q_i(u-b_{i-1})=\Phi_i(u)$, $Q_{1,i}(y)=Q_{1,i}(u-b_{i-1})=\Phi_{1,i}(u)$ to Eqs. (26), (27) lead us to

$$
\begin{aligned}
\Phi_i(u) &= \int_0^{u-b_{i-1}}\Phi_i(u-x)n_i(x)/{c_i}^2dx+G_i(u-b_{i-1})/{c_i}^2,\\
\Phi_{1,i}(u) &= \int_0^{u-b_{i-1}}\Phi_{1,i}(u-x)n_i(x)/{c_i}^2dx+\left(G_i(u-b_{i-1})+G_{1,i}(u-b_{i-1})\right)/{c_i}^2.
\end{aligned}
$$

Then, using Proposition 2 of Chadjicontantinidis and Papaioannou (2009) we get the following.

Theorem 3. *For $u>b_{i-1}$, the functions $\Phi_i(u)$, $\Phi_{1,i}(u)$ for $i=1,\ldots,n$, satisfy the following defective renewal equations*

$$
\begin{aligned}
\Phi_i(u) &= \frac{1}{1+\xi_i}\int_0^{u-b_{i-1}}\Phi_i(u-x)\gamma_i(x)dx+\frac{1}{1+\xi_i}B_i(u), &(28)\\
\Phi_{1,i}(u) &= \frac{1}{1+\xi_i}\int_0^{u-b_{i-1}}\Phi_{1,i}(u-x)\gamma_i(x)dx+\frac{1}{1+\xi_i}\left(B_{1,i}(u)+B_i(u)\right), &(29)
\end{aligned}
$$

where $\gamma_i(u)=(1+\xi_i)n_i(u)/{c_i}^2$, is a proper density function, $B_i(u)=(1+\xi_i)G_i(u-b_{i-1})/{c_i}^2$, $B_{1,i}(u)=(1+\xi_i)G_{1,i}(u-b_{i-1})/{c_i}^2$ and ξ_i is such that $1/(1+\xi_i)=\int_0^\infty z_i(x){c_i}^2dx=1-\frac{(\lambda_1+\lambda_2)\delta+\delta^2}{c_i r_{1i} r_{2i}}<1$. Further if $\delta\to 0^+$ then $\xi_i\to\xi_{i,0}$ such that $\xi_{i,0}=1-\theta_i\big[\lambda(\lambda_2+\lambda_1)m_1+\lambda_2\lambda_1 m_2\big]/(c_i^2 r_{2i}(0))$, if the safety loading factor θ_i is positive.

Now, for $i=1,\ldots,n$ define the associated compound geometric distribution

$$
K_i(x)=1-\overline{K}_i(x)=1-\frac{\xi_i}{1+\xi_i}\sum_{n=0}^{\infty}\left(\frac{1}{1+\xi_i}\right)^n\overline{\Gamma}_i^{\star n}(x),\quad x\geq 0,
$$

where $\overline{\Gamma}_i^{\star n}(x)$ is the tail distribution of the n-fold convolution of $\Gamma_i(x)=1-\overline{\Gamma}_i(x)=\int_0^x\gamma_i(y)dy$. Then, the explicit solutions of the defective renewal equations (28) and (29) can be derived in terms of the associated compound geometric distribution $K_i(x)$.

Theorem 4. *For $u > b_{i-1}$, the functions $\Phi_i(u)$ and $\Phi_{1,i}(u)$ for $i = 1, \ldots, n$, satisfying the defective renewal equations (28) and (29) respectively, can be expressed as*

$$\Phi_i(u) = \frac{1}{\xi_i} \int_0^{u-b_{i-1}} B_i(u-x) dK_i(x) + \frac{1}{1+\xi_i} B_i(u), \tag{30}$$

$$\Phi_{1,i}(u) = \Phi_i(u) + \frac{1}{\xi_i} \int_0^{u-b_{i-1}} B_{1,i}(u-x) dK_i(x) + \frac{1}{1+\xi_i} B_{1,i}(u). \tag{31}$$

Proof. Changing, once again, the variable $y = u - b_{i-1} \geq 0$ to equations (28), (29) and applying Theorem 2.1 of Lin and Willmot (1999) we obtain that

$$Q_i(y) = \frac{1}{\xi_i} \int_0^y B_i(y + b_{i-1} - x) dK_i(x) + \frac{1}{1+\xi_i} B_i(y + b_{i-1}),$$

$$Q_{1,i}(y) = Q_i(y) + \frac{1}{\xi_i} \int_0^y B_{1,i}(y + b_{i-1} - x) dK_i(x) + \frac{1}{1+\xi_i} B_{1,i}(y + b_{i-1}),$$

from which we get the required expressions immediately. □

From Theorem 4, since $B_i(u)$ and $B_{1,i}(u)$ can be easily evaluated for several choices of the penalty function $w(x, y)$, it is clear that $\Phi_i(u)$ and $\Phi_{1,i}(u)$ can be derived explicitly whenever the d.f. $K_i(u)$ is known explicitly. Among these cases are the d.f. $K_i(u)$ that have rational LT $\widehat{K}_i(s)$. Note that $\widehat{K}_i(s)$ is a rational function if and only if $\widehat{f}_1(s)$ and $\widehat{f}_2(s)$ are rational functions.

Thus, we consider the case where the claim size p.d.f. have rational LT, that is

$$\widehat{f}_1(s) = \frac{p_{n-1}(s)}{p_n(s)}, \text{ with } p_{n-1}(0) = p_n(0), \quad \text{and } \widehat{f}_2(s) = \frac{q_{m-1}(s)}{q_m(s)}, \text{ with } q_{m-1}(0) = q_m(0), \tag{32}$$

where $p_{n-1}(s),\ q_{m-1}(s)$ are polynomials of degree $n-1,\ m-1$ or less, respectively and $p_n(s),\ q_m(s)$ are polynomials of degree $n,\ m$ respectively with leading coefficient 1 and with only negative roots. The class of the rational distributions is a wide class of distributions, containing among others the exponential distribution, the Erlang distribution, the Phase-type distribution (as well as the mixture of them).

From Theorem 1 of Chadjiconstantinidis and Papaioannou (2009), the associated compound geometric distribution for claims having LT as in (32) is given by

$$K_i(x) = 1 - \overline{K}_i(x) = 1 - \sum_{j=1}^{2n+m} a_{j,i} e^{-R_{j,i} x}, \quad x \geq 0, \tag{33}$$

where

$$a_{j,i} = \frac{R_{1,i} R_{2,i} \ldots R_{2n+m,i}}{R_{j,i} \prod_{k=1,k\neq j}^{2n+m} (R_{j,i}) - R_{k,i}} \frac{p_n^2(-R_{j,i}) q_m(R_{j,i})}{p_n^2(0) q_m(0)}, \quad j = 1, 2, \ldots, 2n+m.$$

Further $\xi_i/(1+\xi_i) = R_{1,i} R_{2,i} \ldots R_{2n+m,i}/p_n^2(0) q_m(0)$, where $R_{j,i},\ j = 1, 2, \ldots, 2n+m$ are the roots with negative real part of the equation

$$q_m(s) \prod_{k=1}^{2} \left[(c_i s - \lambda - \lambda_k - \delta) p_n(s) + \lambda p_{n-1}(s) \right] - \lambda_1 \lambda_2 q_{m-1}(s) p_n^2(s) = 0. \tag{34}$$

3.2 Analysis of the homogeneous integro-differential equation system

In this subsection, our aim is to solve the homogeneous integro-differential equation system (14) by using LT. A change of variable $y = u - b_{i-1}$ and $\chi_i(y) = \chi_i(u - b_{i-1}) = \upsilon_i(u)$, $\chi_{1,i}(y) = \chi_{1,i}(u - b_{i-1}) = \upsilon_{1,i}(u)$ $y \geq 0$, $i = 1, \ldots, n$, brings (14) into the form

$$\begin{aligned} &c_i{\chi_i}'(y) - (\lambda + \lambda_1 + \delta)\chi_i(y) + \lambda \int_0^y \chi_i(y-x) f_1(x)dx + \lambda_1 \chi_{1,i}(y) = 0, \\ &c_i \chi'_{1,i}(y) + \lambda_2 \int_0^y \chi_i(y-x) f_2(x)dx + \lambda \int_0^y \chi_{1,i}(y-x) f_1(x)dx \\ &\qquad\qquad\qquad\qquad\qquad\qquad - (\lambda + \lambda_2 + \delta)\chi_{1,i}(y) = 0. \end{aligned} \tag{35}$$

The above homogeneous integro-differential equation sysytem has solution of the form

$$\begin{pmatrix} \chi_i(y) \\ \chi_{1,i}(y) \end{pmatrix} = \eta_{i,1} \begin{pmatrix} \chi_{i,11}(y) \\ \chi_{i,21}(y) \end{pmatrix} + \eta_{i,2} \begin{pmatrix} \chi_{i,12}(y) \\ \chi_{i,22}(y) \end{pmatrix}, \quad y \geq 0. \tag{36}$$

Further, from the initial conditions of $\left[\upsilon_{i,11}(u), \upsilon_{i,21}(u)\right]'$, $\left[\upsilon_{i,12}(u), \upsilon_{i,22}(u)\right]'$ we can conclude that $\chi_{i,k\ell}(0) = 1_{(k=\ell)},\ k,\ \ell = 1,2,\ i = 1, 2, \ldots, n$, from which it yields that $\left[\chi_{i,11}(y), \chi_{i,21}(y)\right]'$, $\big[\chi_{i,12}(y),$ $\chi_{i,22}(y)\big]'$ are also some linearly independent solutions.

Let $\widehat{\chi}_i(s) = \int_0^\infty e^{-sy}\chi_i(y)dy,\ \widehat{\chi}_{1,i}(s) = \int_0^\infty e^{-sy}\chi_{1,i}(y)dy$ be the LT of $\chi_i(y)$ and $\chi_{1,i}(y)$ respectively. Then taking LT in both equations of (35) and solving the resulting system of equations we get that

$$\begin{aligned} \widehat{\chi}_i(s) &= \frac{\left[cs - (\lambda_2 + \lambda + \delta) + \lambda \widehat{f}_1(s)\right] c_i\chi_i(0) - c_i\lambda_1\chi_{1,i}(0)}{\gamma_i(s) - \lambda_1\lambda_2\widehat{f}_2(s)}, \\ \widehat{\chi}_{1,i}(s) &= \frac{c_i\chi_{1,i}(0)\left[cs - (\lambda_1 + \lambda + \delta) + \lambda\widehat{f}_1(s)\right] - c_i\lambda_2\widehat{f}_2(s)\chi_i(0)}{\gamma_i(s) - \lambda_1\lambda_2\widehat{f}_2(s)}. \end{aligned}$$

Now, note that from form the above two equations we can get the solutions for $\left[\chi_{i,11}(y), \chi_{i,21}(y)\right]'$, $\left[\chi_{i,12}(y), \chi_{i,22}(y)\right]'$. It is easy to check that t is easy to check that $\chi_i(0) = \eta_1$, $\chi_{1,i}(0) = \eta_2$ and so $\widehat{\chi}_i(s) = \chi_i(0)\widehat{\chi}_{i,11}(s) + \chi_{1,i}(0)\widehat{\chi}_{i,12}(s)$, and $\widehat{\chi}_{1,i}(s) = \chi_i(0)\widehat{\chi}_{i,21}(s) + \chi_{1,i}(0)\widehat{\chi}_{i,22}(s)$ impyling that

$$\begin{aligned} \widehat{\chi}_{i,1\ell}(s) &= \frac{c_i\chi_{i,1\ell}(0)\left[c_is - (\lambda_2 + \lambda + \delta) + \lambda\widehat{f}_1(s)\right] - c_i\lambda_1\chi_{i,2\ell}(0)}{\gamma_i(s) - \lambda_1\lambda_2\widehat{f}_2(s)}, \ell = 1, 2, \\ \widehat{\chi}_{i,2\ell}(s) &= \frac{c_i\chi_{i,2\ell}(0)\left[c_is - (\lambda_1 + \lambda + \delta) + \lambda\widehat{f}_1(s)\right] - c\lambda_2\widehat{f}_2(s)\chi_{i,1\ell}(0)}{\gamma_i(s) - \lambda_1\lambda_2\widehat{f}_2(s)}, \ell = 1, 2 \end{aligned} \tag{37}$$

Using the LT and both Eqs. in (37) we are able to find $\chi_{i,k\ell}(y),\ k, \ell \in \{1, 2\}$, when both claim sizes severities belong to the rational family. Consequently we can determine $\upsilon_{i,k\ell}(u)$, $k, \ell \in \{1, 2\}$ for rational distributed claim sizes as given in the following corollary.

Corollary 1. *If the LT $\widehat{f}_1(s)$ and $\widehat{f}_1(s)$ of the claim densities are defined as in* (32)*, then the two linearly independent solutions $\left[\upsilon_{i,11}(u), \upsilon_{i,21}(u)\right]'$, $\left[\upsilon_{i,12}(u), \upsilon_{i,22}(u)\right]'$, $i = 1, \ldots, n$, are given by*

$$\upsilon_{i,k\ell}(u) = \sum_{j=1}^{2} \bar{a}_{i,k\ell}(j)\, e^{r_{ji}(u - b_{i-1})} + \sum_{j=1}^{2n+m} \bar{b}_{i,k\ell}(j)\, e^{-R_{ji}(u - b_{i-1})}, \quad u > b_{i-1},$$

for $k,\ell \in \{1,2\}$, and

$$\begin{aligned}
\bar{a}_{k\ell}(j) &= p_n^2(r_{ji})\, q_m(r_{ji}) \frac{1_{(k=\ell)}\left[(c_i r_{ji} - \lambda_{kl}^* - \lambda - \delta) + \lambda \frac{p_{n-1}(r_{ji})}{p_n(r_{ji})}\right] - \upsilon_{i,k\ell}^*(b_{i-1}; r_{ji})}{c_i \tau_{i,2}'(r_{ji}) \prod_{k=1}^{2n+m} (r_{ki} + R_{ki})}, j = 1,2, \\
\bar{b}_{k\ell}(i) &= p_n^2(-R_{ji})\, q_m(-R_{ji}) \frac{1_{(k=\ell)}\left[\lambda \frac{p_{n-1}(-R_{ji})}{p_n(-R_{ji})} - (c_i R_{ji} + \lambda_{kl}^* + \lambda + \delta)\right] - \upsilon_{i,k\ell}^*(b_{i-1}; -R_{ji})}{c_i \tau_{i,2}(-R_{ji}) \prod_{k=1,k\neq j}^{2n+m} (R_{ki} - R_{ji})},
\end{aligned}$$

with

$$\lambda_{k\ell}^* = \begin{cases} \lambda_2 & ,k=1 \\ \lambda_1 & ,k=2 \end{cases}, \quad \upsilon_{i,k\ell}^*(b_{i-1}; t) = \begin{cases} \lambda_1 & ,k=1,\ell=2 \\ \lambda_2 \frac{q_{m-1}(t)}{q_m(t)} & ,k=2,\ell=1 \end{cases}$$

where r_{1i}, r_{2i} and R_{ji}, $i = 1,2,...,2n+m$, are the roots of the Eq. (34) and $\tau_{i,2}(s) = (s - r_{1i})(s - r_{2i})$.

Proof. From the form of the LT in (37) and Corollary 3 of Chadjiconstantinidis and Papaioannou (2009) we have that, for $k,\ell \in \{1,2\}$,

$$\chi_{i,k\ell}(y) = \sum_{j=1}^{2} \bar{a}_{i,k\ell}(j)\, e^{r_{ji} y} + \sum_{j=1}^{2n+m} \bar{b}_{i,k\ell}(j)\, e^{-R_{ji} y}, \quad u > b_{i-1},$$

from which we obtain immediately the required equation. □

Remark 2. *Note that $\upsilon_{i,k\ell}(u),\ k,\ell = 1,2$ in Corollary 1 are real-valued functions, even if some of the roots, namely $r_{1i},\ r_{2i},\ -R_{ji},\ j = 1,2,\dots,2n+m$, of the Eq. (34) can come in pairs of complex forms. In this case $\upsilon_{i,k\ell}(u),\ k,\ell = 1,2$ may contain trigonometric functions.*

4 Recursive calculation of the Gerber-Shiu discounted penalty function

In this section, we investigate analytical expressions for the two types of the Gerber-Shiu functions $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ and we show that their explicit recursive expressions can be derived via the results obtained in the previous sections.

From Theorem 4, since the functions $\Phi_i(u)$ and $\Phi_{1,i}(u)$ at each layers $[b_{i-1}, b_i)$ are given in terms of $B_i(u)$ and $B_{1,i}(u)$, from Eqs. (24)-(25) it follows that the functions $B_i(u)$ and $B_{1,i}(u)$ and hence the functions $\Phi_i(u)$ and $\Phi_{1,i}(u)$ can be evaluated if we know the Gerber-Shiu functions $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ at all the previous layers $[0,b_1), [b_1,b_2), \dots, [b_{i-2}, b_{i-1})$. This can be easily achieved by inserting (24)-(25) into Eqs. (30)-(31) and changing the order of the integration, as showing in the following proposition.

Proposition 2. *For $u > b_{i-1}$, $\Phi_i(u)$ and $\Phi_{1,i}(u)$ for $i = 1,\dots,n$, are given by*

$$\Phi_i(u) = \int_0^{b_{i-1}} \phi(y;\mathbf{b}) m_{1,i}(u,y) dy + \int_0^{b_{i-1}} \phi_1(y;\mathbf{b}) m_{2,i}(u,y) dy + m_i(u), \tag{38}$$

$$\Phi_{1,i}(u) = \Phi_i(u) + \int_0^{b_{i-1}} \phi(y;\mathbf{b}) \mu_{1,i}(u,y) dy + \int_0^{b_{i-1}} \phi_1(y;\mathbf{b}) \mu_{2,i}(u,y) dy + \mu_i(u), \tag{39}$$

where

$$\begin{aligned}
m_{k,i}(u,y) &= \frac{1+\xi_i}{\xi_i} \int_0^{u-b_{i-1}} Z_{k,i}(u-x,y) dK_i(x) + Z_{k,i}(u,y), \qquad k = 1,2, \\
m_i(u) &= \frac{1+\xi_i}{\xi_i} \int_0^{u-b_{i-1}} Z_i(u-x) dK_i(x) + Z_i(u),
\end{aligned}$$

and

$$\begin{aligned}\mu_{k,i}(u,y) &= \frac{1+\xi_i}{\xi_i}\int_0^{u-b_{i-1}} H_i(u-x,y)dK_i(x)+H_i(u,y), \qquad k=1,2,\\ \mu_i(u) &= \frac{1+\xi_i}{\xi_i}\int_0^{u-b_{i-1}} H_i(u-x)dK_i(x)+H_i(u).\end{aligned}$$

We remark that formulas (38) and (39) give a recursive approach to calculate $\Phi_i(u)$, $\Phi_{1,i}(u)$ with starting points $\Phi_1(u)=m_1(u)$ and $\Phi_{1,1}(u)=\Phi_1(u)+\mu_1(u)=m_1(u)+\mu_1(u)$, however, the coefficients $\eta_{i,1}$ and $\eta_{i,2}$ can't be explicitly observed. To make more clear, we have the following theorem.

Proposition 3. *For $u>b_{i-1}$, and $i=1,2,\dots,n$ the solutions to non-homogeneous piecewise integro-differential equation system (13) can be expressed as*

$$\Phi_i(u) = L_i(u)+\sum_{k=1}^{i-1}\Big(\eta_{k,1}L_{i,k,1}(u)+\eta_{k,2}L_{i,k,2}(u)\Big), \tag{40}$$

$$\Phi_{1,i}(u) = \Phi_i(u)+\Lambda_i(u)+\sum_{k=1}^{i-1}\Big(\eta_{k,1}\Lambda_{i,k,1}(u)+\eta_{k,2}\Lambda_{i,k,2}(u)\Big), \tag{41}$$

where

$$\begin{aligned}L_i(u) &= m_i(u)+\sum_{k=1}^{i-1}\int_{b_{k-1}}^{b_k}\Big(L_k(y)m_{1,i}(u,y)+\big(L_k(y)+\Lambda_k(y)\big)m_{2,i}(u,y)\Big)dy,\\ \Lambda_i(u) &= \mu_i(u)+\sum_{k=1}^{i-1}\int_{b_{k-1}}^{b_k}\Big(L_k(y)\mu_{1,i}(u,y)+\big(L_k(y)+\Lambda_k(y)\big)\mu_{2,i}(u,y)\Big)dy,\end{aligned}$$

and for $j=1,2$,

$$\begin{aligned}L_{i,k,j}(u) &= \int_{b_{k-1}}^{b_k}\Big(\upsilon_{k,1j}(y)m_{1,i}(u,y)+\upsilon_{k,2j}(y)m_{2,i}(u,y)\Big)dy\\ &\quad+\sum_{\nu=k+1}^{i-1}\int_{b_{\nu-1}}^{b_\nu}\Big(L_{\nu,k,j}(y)m_{1,i}(u,y)+\big(L_{\nu,k,j}(y)+\Lambda_{\nu,k,j}(y)\big)m_{2,i}(u,y)\Big)dy,\\ \Lambda_{i,k,j}(u) &= \int_{b_{k-1}}^{b_k}\Big(\upsilon_{k,1j}(y)\mu_{1,i}(u,y)+\upsilon_{k,2j}(y)\mu_{2,i}(u,y)\Big)dy\\ &\quad+\sum_{\nu=k+1}^{i-1}\int_{b_{\nu-1}}^{b_\nu}\Big(L_{\nu,k,j}(y)\mu_{1,i}(u,y)+\big(L_{\nu,k,j}(y)+\Lambda_{\nu,k,j}(y)\big)\mu_{2,i}(u,y)\Big)dy.\end{aligned}$$

Proof. For $i=1$ the case is trivial. Assume that (40)-(41) still hold for $1\le i\le\nu$. We will prove that Eqs. (40)-(41) are also hold for $i=\nu+1$.

From Eqs (38) and (16), we get that

$$\begin{aligned}\Phi_{\nu+1}(u) &= \int_0^{b_\nu}\phi(y;\mathbf{b})m_{1,\nu+1}(u,y)dy+\int_0^{b_\nu}\phi_1(y;\mathbf{b})m_{2,\nu+1}(u,y)dy+m_{\nu+1}(u)\\ &= m_{\nu+1}(u)+\sum_{i=1}^{\nu}\int_{b_{i-1}}^{b_i}\phi(y;\mathbf{b})m_{1,\nu+1}(u,y)dy+\sum_{i=1}^{\nu}\int_{b_{i-1}}^{b_i}\phi_1(y;\mathbf{b})m_{2,\nu+1}(u,y)dy\\ &= m_{\nu+1}(u)+\sum_{i=1}^{\nu}\int_{b_{i-1}}^{b_i}\Big(\Phi_i(y)+\eta_{i,1}\upsilon_{i,11}(y)+\eta_{i,2}\upsilon_{i,12}(y)\Big)m_{1,\nu+1}(u,y)dy\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\nu}\int_{b_{i-1}}^{b_i}\Big(\Phi_{1,i}(y)+\eta_{i,1}\upsilon_{i,21}(y)+\eta_{i,2}\upsilon_{i,22}(y)\Big)m_{2,\nu+1}(u,y)dy \\
= \quad & m_{\nu+1}(u)+\sum_{i=1}^{\nu}\int_{b_i-1}^{b_i}\Phi_i(y)m_{1,\nu+1}(u,y)dy+\sum_{i=1}^{\nu}\eta_{i,1}\sum_{j=1}^{2}\int_{b_i-1}^{b_i}\upsilon_{i,j1}(y)m_{j,\nu+1}(u,y)dy \\
&+\sum_{i=1}^{\nu}\int_{b_i-1}^{b_i}\Phi_{1,i}(y)m_{2,\nu+1}(u,y)dy+\sum_{i=1}^{\nu}\eta_{i,2}\sum_{j=1}^{2}\int_{b_i-1}^{b_i}\upsilon_{i,j2}(y)m_{j,\nu+1}(u,y)dy, \qquad (42)
\end{aligned}
$$

By assumption, for $1 \le \ell \le \nu$, it holds

$$
\Phi_\ell(u) = L_\ell(u)+\sum_{k=1}^{\ell-1}\big(\eta_{k,1}L_{\ell,k,1}(u)+\eta_{k,2}L_{\ell,k,2}(u)\big),
$$

from which we obtain that

$$
\begin{aligned}
\sum_{\ell=1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}\Phi_\ell(y)m_{1,\nu+1}(u,y)dy \quad &= \quad \sum_{\ell=1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}L_\ell(y)m_{1,\nu+1}(u,y) \\
&\quad +\sum_{\ell=1}^{\nu}\sum_{k=1}^{\ell-1}\sum_{j=1}^{2}\eta_{k,j}\int_{b_{\ell-1}}^{b_\ell}L_{\ell,k,j}(y)m_{1,\nu+1}(u,y)dy \\
&= \quad \sum_{\ell=1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}L_\ell(y)m_{1,\nu+1}(u,y) \\
&\quad +\sum_{k=1}^{\nu-1}\sum_{\ell=k+1}^{\nu}\sum_{j=1}^{2}\eta_{k,j}\int_{b_{\ell-1}}^{b_\ell}L_{\ell,k,j}(y)m_{1,\nu+1}(u,y)dy. \qquad (43)
\end{aligned}
$$

In a similar way, for $1 \le \ell \le n$, one obtains that

$$
\begin{aligned}
\sum_{\ell=1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}\Phi_{1,\ell}(y)m_{2,\nu+1}(u,y)dy \quad &= \quad \sum_{\ell=1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}\big(\Lambda_\ell(y)+L_\ell(y)\big)m_{2,\nu+1}(u,y) \\
&\quad +\sum_{k=1}^{\nu-1}\sum_{\ell=k+1}^{\nu}\sum_{j=1}^{2}\eta_{k,j}\int_{b_{\ell-1}}^{b_\ell}\big(\Lambda_{\ell,k,j}(y)+L_{\ell,k,j}(y)\big)m_{2,\nu+1}(u,y)dy. \qquad (44)
\end{aligned}
$$

Note that in Eqs. (43)-(44) we can relax the constraints $1 \le k \le \nu-1$ to $1 \le k \le \nu$ in the second summation since $\sum_{k=\nu+1}^{\nu} \cdot = 0$. Therefore , inserting Eq. (43) and (44) into Eq. (42) we get that

$$
\begin{aligned}
\Phi_{\nu+1}(u) \quad = \quad & m_{\nu+1}(u)+\sum_{\ell=1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}\Big(L_\ell(y)m_{1,\nu+1}(u,y)+\big(\Lambda_\ell(y)+L_\ell(y)\big)m_{2,\nu+1}(y)\Big)dy \\
&+\sum_{k=1}^{\nu}\eta_{k,1}\Big(\int_{b_{k-1}}^{b_k}\upsilon_{k,11}(y)m_{1,\nu+1}(u,y)+\sum_{\ell=k+1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}L_{\ell,k,1}(y)m_{1,\nu+1}(u,y)dy \\
&+\int_{b_{k-1}}^{b_k}\upsilon_{k,21}(y)m_{2,\nu+1}(u,y)+\sum_{\ell=k+1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}\big(\Lambda_{\ell,k,1}(y)+L_{\ell,k,1}(y)\big)m_{2,\nu+1}(u,y)dy\Big) \\
&+\sum_{k=1}^{\nu}\eta_{k,2}\Big(\int_{b_{k-1}}^{b_k}\upsilon_{k,12}(y)m_{1,\nu+1}(u,y)+\sum_{\ell=k+1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}L_{\ell,k,2}(y)m_{1,\nu+1}(u,y)dy
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{b_{k-1}}^{b_k} \upsilon_{k,22}(y)m_{2,\nu+1}(u,y)+\sum_{\ell=k+1}^{\nu}\int_{b_{\ell-1}}^{b_\ell}\big(\Lambda_{\ell,k,2}(y)+L_{\ell,k,2}(y)\big)m_{2,\nu+1}(u,y)dy\Big)\\
= \quad & L_{\nu+1}(u)+\sum_{k=1}^{\nu}\Big(\eta_{k,1}L_{\nu+1,k,1}(u)+\eta_{k,2}L_{\nu+1,k,2}(u)\Big),
\end{aligned}
$$

which proves (40) for $i=\nu+1$. By using exactly the similar arguments we can also prove that Eq. (41) is also valid for $i=\nu+1$. □

An immediate consequence of the Proposition 3 and Eq. (16) it is that we can calculate the Gerber-Shiu functions $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ explicitly as follows.

Theorem 5. *For $b_{i-1}\le u<b_i$, and for $i=1,2,\ldots,n$, the two types Gerber-Shiu expected discounted penalty functions $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ can be analytically expressed as*

$$
\phi(u;\mathbf{b}) = L_i(u)+\sum_{k=1}^{i-1}\sum_{j=1}^{2}\eta_{k,j}L_{i,k,j}(u)+\sum_{j=1}^{2}\eta_{i,j}\upsilon_{i,1j}(u), \tag{45}
$$

$$
\phi_1(u;\mathbf{b}) = \Lambda_i(u)+\sum_{k=1}^{i-1}\sum_{j=1}^{2}\eta_{k,j}\Lambda_{i,k,j}(u)+\sum_{j=1}^{2}\eta_{i,j}\upsilon_{i,2j}(u). \tag{46}
$$

Now we turn to the determination of the coefficients η_{ij} for $i=1,2,\ldots,n,\ j=1,2$. First note that $\eta_{n,1}=\eta_{n,2}=0$. Recalling the boundary conditions (11) and the initial conditions $\upsilon_{i,k\ell}(b_{i-1})=1_{(k=\ell)},\ k,\ \ell\in\{1,2,\},\ i=1,2,\ldots,n$, one can obtains that $\eta_{i,j}$'s recursively as solutions to the following linear equation system for $m=1,\ldots,n-1$

$$
\begin{aligned}
&c_m\Big(L'_m(b_m)+\sum_{k=1}^{m-1}\sum_{j=1}^{2}\eta_{k,j}L'_{m,k,j}(b_m)+\sum_{j=1}^{2}\eta_{m,j}\upsilon'_{m,1j}(b_m)\Big)=c_{m+1}\Big(L'_{m+1}(b_m)\\
&+\sum_{k=1}^{m}\sum_{j=1}^{2}\eta_{k,j}L'_{m+1,k,j}(b_m)+\sum_{j=1}^{2}\eta_{m+1,j}\upsilon'_{m+1,1j}(b_m)\Big),
\end{aligned}\tag{47}
$$

$$
\begin{aligned}
&c_m\Big(L'_m(b_m)+\sum_{k=1}^{m-1}\sum_{j=1}^{2}\eta_{k,j}L'_{m,k,j}(b_m)+\widetilde{L}'_m(b_m)+\sum_{k=1}^{m-1}\sum_{j=1}^{2}\eta_{k,j}\Lambda'_{m,k,j}(b_m)+\sum_{j=1}^{2}\eta_{m,j}\upsilon'_{m,2j}(b_m)\Big)\\
&=c_{m+1}\Big(L'_{m+1}(b_m)+\sum_{k=1}^{m}\sum_{j=1}^{2}\eta_{k,j}L'_{m+1,k,j}(b_m)+\Lambda'_{m+1,k,j}(b_m)+\sum_{k=1}^{m}\sum_{j=1}^{2}\eta_{k,j}\Lambda'_{m+1,k,j}(b_m)\\
&+\sum_{j=1}^{2}\eta_{m+1,j}\upsilon'_{m+1,2j}(b_m)\Big).
\end{aligned}\tag{48}
$$

Example 1 (The Laplace transform of the time to ruin). Let $\delta>0$ and $w(x,y)\equiv 1$. In that case the Gerber-Shiu functions $\phi(u;\mathbf{b})$ and $\phi_1(u;\mathbf{b})$ are reduced to $\mathbb{E}\big(e^{-\delta T_{\mathbf{b}}}1_{(T_{\mathbf{b}}<\infty)}\big|U_{\mathbf{b}}(0)=u\big)=\varphi_{T_{\mathbf{b}}}(u;\mathbf{b})$ and $\mathbb{E}\big(e^{-\delta(T_{\mathbf{b}}-t)}1_{(T_{\mathbf{b}}<\infty)}\big|U_{\mathbf{b}}(t)=u,L_{11}=t\big)=\varphi_{T_{\mathbf{b}},1}(u;\mathbf{b})$, which it can be seen as the LT of the time to ruin with argument δ. We assume that the claims for both classes are exponentially distributed with densities $f_1(x)=\alpha e^{-\alpha x}$, $\alpha>0$, and $f_2(x)=\beta e^{-\beta x}$, $\beta>0$, $x\ge 0$. Further, we consider the existence of three layers, i.e. $0=b_0<b_1<b_2<b_3=\infty$.

Note that $L_1(u)$ and $L_1+\Lambda_1(u)$ are the LT of the time to ruin in the corresponding two classes risk model with no dividend layers.

Now, consider the following set of parameters: $\alpha = 1, \lambda = 1, \lambda_1 = 0.5, \lambda_2 = 2, \beta = 1.5, \delta = 0.03, c_1 = 15, c_2 = 13, c_3 = 12, b_1 = 3$ and $b_2 = 6$. Then by solving the following equations

$$\begin{aligned}\left(15s - \frac{s}{s+1} - 0.53\right) \cdot \left(15s - \frac{s}{s+1} - 2.03\right) - \frac{1.5}{s+1.5} &= 0,\\ \left(13s - \frac{s}{s+1} - 0.53\right) \cdot \left(13s - \frac{s}{s+1} - 2.03\right) - \frac{1.5}{s+1.5} &= 0,\\ \left(12s - \frac{s}{s+1} - 0.53\right) \cdot \left(12s - \frac{s}{s+1} - 2.03\right) - \frac{1.5}{s+1.5} &= 0,\end{aligned}$$

we obtain that

$$\begin{aligned}&r_{11} = 0.00218, r_{21} = 0.17579, -R_{11} = -0.93081, -R_{21} = -0.94525, -R_{31} = -1.49790,\\ &r_{12} = 0.00256, r_{22} = 0.20389, -R_{12} = -0.91976, -R_{22} = -0.93857, -R_{32} = -1.49734,\\ &r_{13} = 0.00279, r_{23} = 0.22157, -R_{13} = -0.91280, -R_{23} = -0.93459, -R_{32} = -1.49697.\end{aligned}$$

Then from Corollary 1 we obtain that for $u \geq 0$,

$$\begin{aligned}\upsilon_{1,11}(u) &= 0.8736e^{0.00218u} + 0.19595e^{0.17579u} - 0.054533e^{-0.93081u} - 0.01383e^{-0.94525u}\\ &\quad -0.001211e^{-1.49790u},\\ \upsilon_{1,12}(u) &= 0.218468e^{0.00218u} - 0.214243e^{0.17579u} - 0.010742e^{-0.93081u} + 0.006539e^{-0.94525u}\\ &\quad -0.000022e^{-1.49790u},\\ \upsilon_{1,21}(u) &= 0.872604e^{0.00218u} - 0.767079e^{0.17579u} - 0.1132e^{-0.93081u} + 0.070725e^{-0.94525u}\\ &\quad -0.063013e^{-1.49790u},\\ \upsilon_{1,22}(u) &= 0.218214e^{0.00218u} + 0.838682e^{0.17579u} - 0.022306e^{-0.93081u} - 0.033445e^{-0.94525u}\\ &\quad -0.001145e^{-1.49790u},\end{aligned}$$

and for $u \geq 3$

$$\begin{aligned}\upsilon_{2,11}(u) &= 0.879333e^{0.00256u} + 0.105739e^{0.20389u} - 1.016948e^{-0.91976u} - 0.252484e^{-0.93857u}\\ &\quad -0.134526e^{-1.49734u},\\ \upsilon_{2,12}(u) &= 0.219908e^{0.00256u} - 0.117179e^{0.20389u} - 0.201443e^{-0.919761u} + 0.120413e^{-0.93857u}\\ &\quad -0.002745e^{-1.49734u},\\ \upsilon_{2,21}(u) &= 0.878136e^{0.00256u} - 0.412631e^{0.20389u} - 2.083034e^{-0.91976u} + 1.286867e^{-0.93857u}\\ &\quad -6.189831e^{-1.49734u},\\ \upsilon_{2,22}(u) &= 0.219609e^{0.00256u} + 0.457277e^{0.20389u} - 0.412619e^{-0.91976u} - 0.613724e^{-0.93857u}\\ &\quad +0.126291e^{-1.49734u},\end{aligned}$$

Moreover, substituting the parameters into the functions of Proposition 3

$$\begin{aligned}L_1(u) &= 0.060096e^{-0.93081u} + 0.007318e^{-0.94525u} + 0.001145e^{-1.49790u},\\ \Lambda_1(u) + L_1(u) &= 0.12479e^{-0.93081u} - 0.03743e^{-0.94525u} + 0.059755e^{-1.49790u},\end{aligned}$$

and thus by the recursive formulas of Proposition 3, we obtain that

$$\begin{aligned}L_2(u) &= 0.06765e^{-0.91976u} + 0.007864e^{-0.93857u} + 0.00122e^{-1.49734u},\\ \Lambda_2(u) &= 0.12071e^{-0.91976u} - 0.02561e^{-0.93857u} + 0.056339e^{-1.49734u},\\ L_3(u) &= 0.0705e^{-0.91280u} + 0.00818e^{-0.93459u} + 0.00049e^{-1.49697u},\\ \Lambda_3(u) &= 0.11173e^{-0.91280u} - 0.0147e^{-0.93459u} + 0.022188e^{-1.49697u},\end{aligned}$$

$$
\begin{aligned}
L_{2,1,1}(u) &= 0.97682e^{-0.91976u} + 0.33765e^{-0.93857u} + 0.14899e^{-1.49734u}, \\
L_{2,1,2}(u) &= 0.15719e^{-0.91976u} - 0.22191e^{-0.93857u} - 0.015282e^{-1.49734u}, \\
\Lambda_{2,1,1}(u) &= 2.23455e^{-0.91976u} - 1.91026e^{-0.93857u} + 6.854349e^{-1.49734u}, \\
\Lambda_{2,1,2}(u) &= -0.950203e^{-0.91976u} + 2.161659e^{-0.93857u} - 0.69628e^{-1.49734u},
\end{aligned}
$$

and

$$
\begin{aligned}
L_{3,1,1}(u) &= 1.02174e^{-0.91280u} + 0.34799e^{-0.93459u} + 0.15425e^{-1.49697u}, \\
L_{3,1,2}(u) &= 0.154412e^{-0.91280u} - 0.22267e^{-0.93459u} - 0.01699e^{-1.49697u}, \\
L_{3,2,1}(u) &= -0.31089e^{-0.91280u} + 0.95295e^{-0.93459u} + 0.059362e^{-1.49697u}, \\
L_{3,2,2}(u) &= 4.27725e^{-0.91280u} - 3.63864e^{-0.93459u} - 0.16198e^{-1.496968u}, \\
\Lambda_{3,1,1}(u) &= 2.09744e^{-0.91280u} - 1.78902e^{-0.93459u} + 6.63376e^{-1.49697u}, \\
\Lambda_{3,1,2}(u) &= -0.89626e^{-0.91280u} + 2.16707e^{-0.93459u} - 0.68497e^{-1.49697u}, \\
\Lambda_{3,2,1}(u) &= 37.9558e^{-0.91280u} - 34.8825e^{-0.93459u} + 657.6117e^{-1.49697u}, \\
\Lambda_{3,2,2}(u) &= -17.2047e^{-0.91280u} + 40.31738e^{-0.93459u} - 79.0007e^{-1.49697u},
\end{aligned}
$$

Finally from Eqs. (47) and (48), we obtain the following linear equation system

$$
\begin{pmatrix}
1.981633 & -0.992642 & -0.117692 & 0.038461 \\
-1.553575 & 4.509101 & 0.4 \times 10^{-9} & -0.233077 \\
-0.000498 & 0.000071 & 1.067405 & -0.997872 \\
-0.002958 & -0.0006476 & -1.84071 & 4.86628
\end{pmatrix}
\begin{pmatrix}
\eta_{1,1} \\ \eta_{1,2} \\ \eta_{2,1} \\ \eta_{2,2}
\end{pmatrix}
=
\begin{pmatrix}
0.000542 \\ 0.005633 \\ 0.000024 \\ 0.000245
\end{pmatrix},
$$

from which we get that

$$
\begin{aligned}
\eta_{1,1} &= 0.001096, \quad \eta_{1,2} = 0.001632, \\
\eta_{2,1} &= 0.000109, \quad \eta_{2,2} = 0.0000927.
\end{aligned}
$$

Hence, from Theorem 5, $\varphi_T(u;\mathbf{b})$ and $\varphi_{T,1}(u;\mathbf{b})$ are given by

$$
\varphi_{T_{\mathbf{b}}}(u;\mathbf{b}) = \begin{cases}
0.06002e^{-0.93081u} + 0.007314e^{-0.94525u} + 0.001147e^{-1.49790u} & \\
+0.001314e^{0.00218u} - 0.0001349e^{0.17579u}, & 0 \le u < 3, \\
0.068851e^{-0.91976u} + 0.007855e^{-0.93857u} + 0.001346e^{-1.49734u} & \\
+0.0001167e^{0.00256u} + 0.7129 \times 10^{-6}e^{0.20389u}, & 3 \le u < 6, \\
0.072235e^{-0.91280u} + 0.00797e^{-0.93459u} + 0.000621e^{-1.49697u} & u \ge 6
\end{cases}
$$

and

$$
\varphi_{T_{\mathbf{b}},1}(u;\mathbf{b}) = \begin{cases}
0.12463e^{-0.93081u} + 0.007314e^{-0.94525u} + 0.059684e^{-1.49790u} & \\
+0.0013121e^{0.00218u} + 0.000528e^{0.17579u}, & 0 \le u < 3, \\
-0.02409e^{-0.91976u} + 0.12134e^{-0.93857u} + 0.06202e^{-1.49734u} & \\
+0.0001165e^{0.00256u} - 0.278238 \times 10^{-5}e^{0.20389u}, & 3 \le u < 6, \\
0.115128e^{-0.91280u} - 0.013249e^{-0.93459u} + 0.0093041e^{-1.49697u} & u \ge 6
\end{cases}
$$

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