Finite time ruin probabilities for Phase-type claims

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Notation for the Sparre Andersen (SA) risk model

The surplus process is

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

where u is the initial reserve, c is the premium income rate, $N(t)$ the number of claims up to time t, and X_i, $i = 1, 2, \ldots,$ are the i.i.d. claims.

Claim interarrival times A_i are also i.i.d. in the (ordinary) renewal risk model, while if A_1 has a different distribution from $A_j, j \geq 2$, we have the delayed renewal risk model.

Let $\tau(u)$ be the time to ruin, when the initial reserve is u, i.e.

$$\tau(u) = \inf\{t > 0 : \ R(t) < 0 | R(0) = u\}$$

Setup and Assumptions

We assume that the claim sizes X_j are phase-type distributed with representation $(\boldsymbol{\alpha}, \mathbf{T})$, density f and cdf F.

Let $\mathbf{t}$ be a column vector with the intensities to the absorption state from each transient state,

$$\mathbf{t} = -\mathbf{T}\mathbf{1},$$

where $\mathbf{1}$ is a column vector of ones. Then for $\overline{F}(x) = 1 - F(x)$, we have

$$\overline{F}(x) = \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{1} \qquad \text{and} \qquad f(x) = \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{t}.$$

Let H be the distribution of A_j, $j \geq 2$, and let H_1 be the distribution function of A_1, which may differ from H.

Several papers deal with the distribution of the time to ruin or its Laplace transform in the case where both the interarrival times and the claim amounts are phase-type.

- Badescu et al. (2005), Stanford et al. (2005), Borovkov and Dickson (2008), Dickson and Li (2010)

Much of this research was initiated by the introduction of the Gerber - Shiu function,

$$m_\delta(u) = \mathbb{E}(e^{-\delta\tau(u)} w\left(R(\tau(u)-), |R(\tau(u))|\right) I(\tau(u) < \infty)),$$

where $\delta > 0$, w is a function $\mathbb{R} \to \mathbb{R}$, and $I(A)$ is 1 if the event A occurs and 0 otherwise.

Here we focus instead on the function

$$\mathbb{E}(e^{-\delta\tau(u)} z^{N_u} w(|R(\tau(u))|) I(\tau(u) < \infty)),$$

where $0 < z < 1$ and N_u is **number of claims until ruin** with initial capital u,

$$N_u = \inf\{n : \ u + \sum_{i=1}^{n} (cA_i - X_i) < 0\}$$

For $u = 0$, we write N instead of N_0.

Let $J(t)$ be the Markov process that describes the phase of the claims at time t.

Then we define a matrix $\mathbf{G}(z, s)$ as follows:

$$\mathbf{G}_{i,j}(z, s) = \mathbb{E}[e^{-s\tau(0)} z^N I(J(\tau(0)) = j) | J(0) = i].$$

In the context of queuing theory, it was proved by Lucantoni (1991) (see also Zhu and Prabhu(1991)) that

$$\mathbf{G}(z, s) = z \int_0^\infty \exp[-(s\mathbf{I} - \mathbf{T} - \mathbf{t}\boldsymbol{\alpha}\mathbf{G}(z, s))x] dH(x),$$

where H is the distribution function of the time between claims.

Main Result

Theorem

Consider a risk process with phase-type claim size distribution as described above, and interarrival distribution H, premium rate c, and initial reserve u. The time until the first arrival has distribution H_1.

Let $\tau(u)$ be the time to ruin, and N_u the number of claims until ruin occurs.

Then

$$\begin{aligned}&\mathbb{E}[e^{-\delta\tau(u)}z^{N_u}w(|R(\tau(u))|)I(\tau(u)<\infty]\\&=\text{exp}(\delta u/c)(\boldsymbol{\alpha}\mathbf{G}_1(z,\delta/c)\\&\cdot\text{exp}(-(\delta\mathbf{I}/c-\mathbf{T}-\mathbf{t}\boldsymbol{\alpha}\mathbf{G}(z,\delta/c))u)\int_0^\infty w(y)\,\text{exp}(\mathbf{T}y)\mathbf{t}dy,\end{aligned}$$

provided that the integral exists, where

$$\mathbf{G}_1(z,\delta/c)=z/c\int_0^\infty \text{exp}[-(\delta\mathbf{I}/c-\mathbf{T}-\mathbf{t}\boldsymbol{\alpha}\mathbf{G}(z,\delta/c))x]dH_1(x/c).$$

Substituting $z = 1$ and $w(y) = 1$ in the theorem we obtain the Laplace transform of the time to ruin for initial reserve u and $H_1 = H$:

$$\mathbb{E}[e^{-\delta\tau(u)}I(\tau(u) < \infty)] = \exp(\delta u/c)(\boldsymbol{\alpha}\mathbf{G}(1, \delta/c) \\ \cdot \exp(-(\delta\mathbf{I}/c - \mathbf{T} - \mathbf{t}\boldsymbol{\alpha}\mathbf{G}(1, \delta/c))u)\mathbf{1},$$

where

$$\mathbf{G}(1, s) = 1/c \int_0^\infty \exp[-(s\mathbf{I} - \mathbf{T} - \mathbf{t}\boldsymbol{\alpha}\mathbf{G}(1, s))x]dH(x/c)dx\,.$$

Thus the (i, j) element of $\mathbf{G}(1, \delta/c)$ is the Laplace transform of the time to ruin when the initial reserve is 0, the first phase of the claim size is i, and the claim causing ruin is at phase j when ruin occurs.

Further, putting $\delta = 0$ in the theorem we obtain that the probability generating function (pgf) of the number of claims until ruin for initial reserve u and $H_1 = H$ is:

$$\mathbb{E}[z^{N_u} I(\tau(u) < \infty)] = \boldsymbol{\alpha}\mathbf{G}(z, 0) \exp[(\mathbf{T} + \mathbf{t}\boldsymbol{\alpha}\mathbf{G}(z, 0)u)]\mathbf{1}.$$

Let $\mathbf{G}(s) = \mathbf{G}(1, s)$ and put $\boldsymbol{\alpha}_+(s) = \boldsymbol{\alpha}\mathbf{G}(s)$, then $\boldsymbol{\alpha}_+(s)$ satisfies the following matrix equation:

$$\boldsymbol{\alpha}_+(s) = \boldsymbol{\alpha} \int_0^\infty \exp[-(s\mathbf{I} - \mathbf{T} - \mathbf{t}\boldsymbol{\alpha}_+(s))x] dH(x),$$

and the ruin probability is $\psi(u) = \boldsymbol{\alpha}_+(0) \exp((\mathbf{T} + \mathbf{t}\boldsymbol{\alpha}_+(0))u)\mathbf{1}$.The ith element of $\boldsymbol{\alpha}_+(0)$ is the probability of ruin when the initial reserve is 0, and ruin occurs when the claim causing ruin is at phase i. The ruin probability for $u = 0$ is simply $\boldsymbol{\alpha}_+(0)\mathbf{1}$. The distribution of the deficit at ruin, given that ruin occurs is phase-type with representation

$$(\boldsymbol{\alpha}_+(0) \exp((\mathbf{T} + \mathbf{t}\boldsymbol{\alpha}_+(0))u)/\psi(u), \mathbf{T}).$$

This result is obtained by considering the excess time at u of the terminating phase-type renewal process where the inter-arrival times are phase-type with representation $(\boldsymbol{\alpha}_+(0), \mathbf{T})$.

The number of claims until ruin

The Panjer class of distributions

Assume that a (discrete) rv N is such that $p_n = \mathbb{P}(N = n) = 0$ for all $n = 0, 1, 2, \ldots, m-1$, while for $n \geq m+1$ the following recursive relationship holds

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}.$$

The class of distributions satisfying the above is the Panjer (a, b, m) class.

For $m = 1$, it is well-known that this class contains

- the binomial distribution
- the Poisson distribution
- the negative binomial distribution

and

- the Extended Negative Binomial (ENB) distribution. We say that N has an ENB distribution with parameters (θ, α) if

$$\begin{aligned} p_n &= \frac{\binom{\alpha+n-1}{n}\theta^n}{(1-\theta)^{-\alpha}-1} \\ &= \frac{-\Gamma(\alpha+n)\theta^n}{n!\Gamma(\alpha)\left[1-(1-\theta)^{-\alpha}\right]} \end{aligned}$$

for $n = 1, 2, \ldots$, while $p_0 = 0$.

Note that the range of the parameter values are $0 < \theta \leq 1$ and $-1 < \alpha < 0$.

Exponential interarrival times and claim sizes

In the SA model, assume that

- claims $X_1, X_2, \ldots$ are iid with an exponential density, $f(x) = \beta e^{-\beta x}$,
- interclaim times $A_1, A_2, \ldots,$ are iid with density $h(x) = \lambda e^{-\lambda x}$

Assume that the initial capital $u = 0$. Then we have the following:

(i) The defective distribution of the number of claims until ruin, N, is given by

$$\mathbb{P}(N = k) = \frac{2^{2k-2}\lambda^k(\beta c)^{k-1}\Gamma(k - 1/2)}{(\beta c + \lambda)^{2k-1}k!\Gamma(1/2)}$$

for $k = 1, 2, 3, \ldots$.

This can be written in the apparently simpler form, which avoids in particular the use of the Gamma function, as follows

$$\mathbb{P}(N = k) = \begin{cases} \dfrac{\lambda}{\beta c + \lambda} & k = 1 \\ \\ \dfrac{2\lambda^k (\beta c)^{k-1} (2k-3)!}{(\beta c + \lambda)^{2k-1} k! (k-2)!} & k = 2, 3, \ldots \end{cases}$$

(ii) The (proper) distribution of the random variable $\tilde{N} = N | N < \infty$ is $ENB(\theta, \alpha)$ with parameters

$$\theta = \frac{4\lambda\beta c}{(\beta c + \lambda)^2}, \qquad \alpha = -\frac{1}{2}.$$

Remarks

- It is interesting that the number of steps until the first entry into $(-\infty, 0)$, say $N^{(\mathbf{s})}$, of the simple random walk with step distribution

$$P(X = 1) = p = 1 - P(X = -1)$$

has also an ENB distribution.

- For exponential claim amounts, the defict at ruin is independent of the number of claims until ruin (whatever the interarrival distribution).

- The case $u > 0$

In the case where the claim amounts are exponentially distributed, putting $\delta = 0$ and $w = 1$ in the theorem we obtain that $P(z; u) = \mathbf{G}(z, u)$, and this gives

$$P(z; u) = P(z; 0) \exp[(-\beta + \beta P(z; 0))u].$$

Thus the derivative at $z = 0$ can be taken iteratively.

Erlang interarrival times

Consider the Sparre Andersen model where claims $X_1, X_2, \ldots$ are iid with an exponential density, $f(x) = \beta e^{-\beta x}$, and interclaim times $A_1, A_2, \ldots,$ are iid with density $h(x)$ which is Gamma (n, μ), i.e.

$$h(x) = \frac{\mu^n x^{n-1} e^{-\mu x}}{(n-1)!}.$$

Assume that the premium rate is $c > 0$ and the initial capital $u = 0$. Then we have the following:

The distribution of the number of claims until ruin, N, is given by

$$\mathbb{P}(N = k) = \frac{\mu^{nk} (\beta c)^{k-1} \left[(n+1)k - 2\right]!}{(\beta c + \mu)^{(n+1)k-1} k!(nk-1)!}$$

for $k = 1, 2, 3, \ldots$.

This distribution is again defective. Divided by $\psi(0)$, so that this becomes proper, the distribution of $\tilde{N}$ is no longer in the ENB class. Instead, it satisfies the recursion

$$p_k = \frac{\sum_{i=0}^{n} a_i k^i}{\sum_{j=0}^{n} b_j k^j} p_{k-1}$$

for $k = 2, 3, \ldots$.

- Thus the ratio of probabilities p_k/p_{k-1} can be written as a ratio of two polynomials of order n (the shape parameter of the Erlang distribution).

- This class was studied by Panjer and Willmot (1982) and Hesselager (1994) (when the recursion is valid for $k \geq 1$).

Example

We use an example from Thorin and Wikstad (1973). The claim amounts are exponentially distributed with mean 1, and the interclaim time has a Pareto distribution

$$H(t) = 1 - (1 + 2t)^{-3/2}.$$

In this case $\alpha_+(\delta)$ is the Laplace transform of the time to ruin when the initial reserve is 0,

$$\alpha_+(\delta) = \int_0^\infty \exp(-(\delta/c + 1 - \alpha_+(\delta)x)(3/c)(1 + 2x/c)^{-2.5}dx.$$

We solved this equation iteratively and then applied the Gaver-Stehfest Laplace inversion algorithm to find the probability of ruin in finite time.

Finite time ruin probabilities: Pareto inter-arrival times and Exponential claims, $c = 1.1$

	u	$\psi(u,t)$	$\psi^{TW}(u,t)$
$t = 100$	0	0.97724	0.97739
	100	0.00101	0.00125
	1000	0.00000	0.00000
$t = 1000$	0	0.99129	0.99129
	100	0.32873	0.32876
	1000	0.00000	0.00000
$t = 10000$	0	0.99403	0.99439
	100	0.55105	0.56403
	1000	0.00077	0.00076
	10000	0.00000	0.00000
$t = \infty$	0	0.99460	0.99460
	100	0.57975	0.57976
	1000	0.00450	0.00450
	10000	0.00000	0.00000

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