Characterization through Hazard Rate of heavy tailed distributions and some Convolution Closure Properties

Anastasios Bardoutsos, Dimitrios Konstantinides

Department of Statistics and Actuarial - Financial Mathematics, University of the Aegean

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INTRODUCTION

We will assume for the whole paper that the distribution function $F$ is supported on $[0, \infty)$ and that $F$ has positive Lebesgue density function $f$.
We use the Matuszeweska indices and its properties to show asymptotics inequalities for the hazard rates. We discuss about the relation membership in dominatedly or subversively varying tail distribution and a hazard rate condition. Convolution closure is establish for the class of distributions with subexponential and subversiverly varying tails.
The class $\mathcal{L}$

A distribution function $F$ belongs to the class $\mathcal{L}$ if

$$\lim_{x \to \infty} \frac{\bar{F}(x - y)}{\bar{F}(x)} = 1$$

for all constants $y \in \mathbb{R}$.

This distribution function $F$ is said to have a long tail.
The class $\mathcal{D}$

A distribution function $F$ belongs to the class $\mathcal{D}$ if

$$\limsup_{x \to \infty} \frac{F(ux)}{F(x)} < \infty$$

(1)

for any $0 < u < 1$ (or equivalently for $u = 1/2$). Such a distribution function $F$ is said to have a dominatedly varying tail. An equivalent way to write (1) is

$$\bar{F}_*(u) := \liminf_{x \to \infty} \frac{\bar{F}(ux)}{\bar{F}(x)} > 0$$

(2)

for any $u > 1$. 

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The class $\mathcal{E}$

A distribution function $F$ belongs to the class $\mathcal{E}$, if for some $u > 1$

$$\bar{F}^*(u) := \limsup_{x \to \infty} \frac{F(ux)}{F(x)} < 1.$$ 

Such a distribution function $F$ is said to have subversively varying tail.
The class $S$

A distribution function $F$ belongs to the class $S$ if

$$\lim_{x \to \infty} \frac{F^*n(x)}{F(x)} = n$$

for any $n \geq 2$ (or equivalently for $n = 2$), where $F^*n$ denotes the $n$th convolution of $F$. Such a distribution function $F$ is said to have a subexponential tail.
The class $\mathcal{A}$

A distribution function $F$ belongs to the class $\mathcal{A}$ if

$$F \in S \text{ and } \bar{F}^*(u) < 1,$$

for some $u > 1$. In other words $\mathcal{A} = \mathcal{E} \cap S$. 
It is well known the following inclusions

\[ D \cap A \subset D \cap L \subset S \subset L. \]

\[ D \cap A \subset D \cap E \]
Definition of Matuszewska indices

Upper Matuszewska index

Let $g$ be positive. Then

The upper Matuszewska index $\alpha(g)$ is the infimum of those $\alpha$ for which there exists a constant $C = C(\alpha) > 0$ such that for each $\Lambda > 1$

$$\frac{g(\lambda x)}{g(x)} \leq C(1 + o(1))\lambda^\alpha,$$

as $x \to \infty$ and uniformly in $\lambda \in [1, \Lambda]$. 
Lower Matuszewska index

Let $g$ be positive. Then

The lower Matuszewska index $\beta(g)$ is the supremum of those $\beta$ for which there exists a constant $D = D(\beta) > 0$ such that for each $\Lambda > 1$

$$\frac{g(\lambda x)}{g(x)} \geq D(1 + o(1))\lambda^\beta,$$

as $x \to \infty$, and uniformly in $\lambda \in [1, \Lambda]$. 
Let consider the following notation. The positive function $g$ has

1. bounded increase, $(g \in BI)$, if $\alpha(g) < \infty$
2. bounded decrease, $(g \in BD)$, if $\beta(g) > -\infty$
3. positive increase, $(g \in PI)$, if $\beta(g) > 0$
4. positive decrease, $(g \in PD)$, if $\alpha(g) < 0$

An equation that holds between the Matuszewska indices is

$$\beta(g) = -\alpha \left( \frac{1}{g} \right), \quad \text{for } g \text{ positive} \quad (3)$$

For more details of the Matuszewska indices, see Chapter 2.1 of Bingham et al. (1987).
Potter Inequalities

Let $g(.)$ be positive.

If $g \in BI$ then for every $\alpha > \alpha(g)$ there exist positive constants $C, x_0$ such that

$$\frac{g(y)}{g(x)} \leq C \left( \frac{y}{x} \right)^{\alpha}, \quad (y \geq x \geq x_0) \quad (4)$$

If $g \in BD$ then for every $\beta < \beta(g)$ there exist positive constants $C', x_0$ such that

$$\frac{g(y)}{g(x)} \geq C' \left( \frac{y}{x} \right)^{\beta}, \quad (y \geq x \geq x_0) \quad (5)$$
Matuszewska indices for the distributions tails


The upper Matuszewska index $\gamma_F$ for a distribution function $F$, was introduced as follows.

$$\gamma_F := \inf \left\{ -\frac{\log F_*(u)}{\log u} : u > 1 \right\} = - \lim_{u \to \infty} \frac{\log F_*(u)}{\log u}. \quad (6)$$

The lower Matuszewska index $\delta_F$ for a distribution function $F$, was introduced as follows.

$$\delta_F := \sup \left\{ -\frac{\log F_*(u)}{\log u} : u > 1 \right\} = - \lim_{u \to \infty} \frac{\log F_*(u)}{\log u}. \quad (7)$$
Inequalities for the distributions functions

If the Upper Matuszeska index $\gamma_F < \infty$ then there exist constants $C, x_0$ such that

$$\frac{\overline{F}(x)}{\overline{F}(y)} \leq C \left(\frac{x}{y}\right)^{-\gamma}$$

(8)

for every $x \geq y \geq x_0$ and $\gamma_F < \gamma < \infty$.

If the Lower Matuszeska index $\delta_F > 0$ is finite then there exist constants $C', x_0$ such that

$$\frac{\overline{F}(x)}{\overline{F}(y)} \geq C' \left(\frac{x}{y}\right)^{-\delta}$$

(9)

for every $x \geq y \geq x_0$ and $0 < \delta < \delta_F$. 

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Matuszewska indices for the density functions

We introduced the upper Matuszewska index $\gamma_f$ for the density function, as follows

$$\gamma_f := \inf \left\{ -\frac{\log f_*(u)}{\log u} : u > 1 \right\} = -\lim_{u \to \infty} \frac{\log f_*(u)}{\log u}. \quad (10)$$

where $f_*(u) = \liminf_{x \to \infty} \frac{f(ux)}{f(x)}$.

We introduced the lower Matuszewska index $\delta_f$ for the density function, as follows

$$\delta_f := \sup \left\{ -\frac{\log f_*(u)}{\log u} : u > 1 \right\} = -\lim_{u \to \infty} \frac{\log f_*(u)}{\log u}. \quad (11)$$

where $f_*(u) = \limsup_{x \to \infty} \frac{f(ux)}{f(x)}$. 

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**Definition**

We will say that the density function has

1. **bounded decrease**, \((f \in BD)\), if \(\gamma_f = \alpha(f^{-1}) < \infty\)

2. **positive decrease**, \((f \in PD)\), if \(\delta_f = \beta(f^{-1}) > 0\)
In analogy to inequalities (8) and (9) we introduce inequalities for density function

**Corollary**

If \( f \in BD \) then there exist constants \( C, x_0 \) such that

\[
\frac{f(y)}{f(x)} \geq C' \left( \frac{y}{x} \right)^{-\gamma}
\]

for every \( y \geq x \geq x_0 \).

If \( f \in PD \) then there exist constants \( C', x_0 \) such that

\[
\frac{f(y)}{f(x)} \leq C \left( \frac{y}{x} \right)^{-\delta}
\]

for every \( y \geq x \geq x_0 \).
Subversively Varying Tail

The Subversively Class is a large class that extends out of the class of Heavy Tail. We can see the following Example

Example

The Exponential distribution function is \( \bar{F}(x) = \exp\{-\lambda x\} \). As we can see

\[
\lim_{x \to \infty} \frac{\bar{F}(ux)}{\bar{F}(x)} = 0 < 1
\]

for all \( u > 1 \).
Lemma (Konstantinides et.al 2002)

Let $F$ be a d.f. with a density function $f$ which is eventually non-increasing. Then the following statements are equivalent:

1. $\bar{F}^*(u) < 1$ holds for some $u > 1$;
2. $\bar{F}^*(u) < 1$ holds for any $u > 1$;
3. the hazard rate function of $F$, $h(x) = \frac{f(x)}{\bar{F}(x)}$ satisfies

$$\liminf_{x \to \infty} xh(x) > 0.$$
Eventually non increasing?

If a function $g$ is eventually non increasing then

$$\forall y \geq x \geq x_0 \Rightarrow g(y) \leq g(x)$$  \hspace{1cm} (14)
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- Does the condition (14) hold for every function $g$?
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We can obtain that condition (14) is the Potter inequality (4) for $C \leq 1$ and $\alpha(g) < 0$. Obviously
Eventually non increasing?

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$$\forall y \geq x \geq x_0 \Rightarrow g(y) \leq g(x) \tag{14}$$

- How easy we check that condition (14) holds?
- Does the condition (14) hold for every function $g$?

We can obtain that condition (14) is the Potter inequality (4) for $C \leq 1$ and $\alpha(g) < 0$. Obviously

$$\frac{g(y)}{g(x)} \leq C \left(\frac{y}{x}\right)^{\alpha} \leq C, \quad (x \geq y \geq x_0). \tag{15}$$

for all $\alpha(g) < \alpha < 0$. But to obtain, if the Potter inequality holds we only need to calculate the Upper Matuszewska index.
The main idea is to replace the condition of eventually non-increasing (or decreasing) with the Potter Inequalities. When we have density functions we can obtain an analogous inequality of the previous one, if the $\delta_f > 0$. 
Let $F$ be an absolute continuous distribution function supported on $[0, \infty)$ with a density function $f(x)$ with $\delta_f > 0 \ (f \in PD)$. Then

$$F \in \mathcal{E} \text{ if and only if } \liminf_{x \to \infty} x h(x) > 0.$$  

(16)
A sufficient condition for \( F \in \mathcal{E} \) is given from the following theorem:

**Theorem**

\[
\text{If } \liminf_{x \to \infty} x h(x) > 0 \text{ then } F \in \mathcal{E}
\]

A similar theorem is found in Klüppelberg C. (1988) for the class \( \mathcal{D} \cap \mathcal{L} \).
Lemma

If \( f \in PD, \delta_f > 1 \) for any \( \delta \in (1, \delta_f) \) then:

\[
xh(x) = x \frac{f(x)}{F(x)} \geq \frac{(\delta - 1)}{C} > 0
\]

for all \( x \geq x_0 \) and \( C > 0 \).

From Lemma and previous theorem we obtain
**Lemma**

If $f \in PD$, $\delta_f > 1$ for any $\delta \in (1, \delta_f)$ then:

$$xh(x) = x \frac{f(x)}{F(x)} \geq \frac{\delta - 1}{C} > 0$$

for all $x \geq x_0$ and $C > 0$.

From Lemma and previous theorem we obtain

**Corollary**

If $f \in PD$ with $\delta_f > 1$ then $F \in \mathcal{E}$. 
Theorem

If $F_1, F_2 \in \mathcal{E}$ and the following statements hold

1. $f_1 \in PD$
2. $x^{-\delta^*} = O\left(\frac{1}{F_1(x)}\right)$

where $\delta^* = \min(\delta_{F_1}, \delta_{F_2})$, then $F_1 \ast F_2 \in \mathcal{E}$. 
Subexponential Class

A sufficient condition for $F \in \mathcal{D} \cap \mathcal{L}$ is given by the following theorem.

**Theorem (Klüppelberg C. 1988)**

If $\limsup_{x \to \infty} x h(x) < \infty$ then $F \in \mathcal{D} \cap \mathcal{L}$

**Corollary (Klüppelberg C. 1988)**

Let $F$ have an eventually decreasing density $f$. Then the following statements are equivalent:

1. $F \in \mathcal{D}$
2. $F \in \mathcal{D} \cap \mathcal{L}$
3. $\limsup_{x \to \infty} x h(x) < \infty$
**Theorem**

Let $F$ be an absolute continuous distribution function supported on $[0, \infty)$ with a density function $f(x)$ with $\delta_f > 0$ ($f \in PD$). Then

1. $F \in \mathcal{D}$
2. $F \in \mathcal{D} \cap \mathcal{L}$
3. $\limsup_{x \to \infty} xh(x) < \infty$
Lemma

If \( f \in BD \) then there is positive \( x_0 \), such that for all \( x \geq x_0 \) and all \( \lambda > 1 \):

\[
xh(x) = x \frac{f(x)}{F(x)} \leq \frac{-\gamma + 1}{C'(\lambda^{-\gamma+1} - 1)}
\]

Furthermore if \( f \in BD \) then \( F \in D \cap L \).
The class \( A \cap D \)

**Theorem**

Let \( F \) be an absolute continuous distribution function with a density function \( f(x) \) with \( \delta_f > 0 \) then \( F \in A \cap D \) if and only if one of the following statements holds

1. \( 0 < \liminf_{x \to \infty} xh(x) \leq \limsup_{x \to \infty} xh(x) < \infty \)
2. \( 0 < \overline{F}_*(u) \leq \overline{F}^*(u) < 1 \)
Pitman (A Characterization Theorem for $S$)

**Theorem (Pitman(1980))**

Suppose $F$ is absolutely continuous with density function $f$ and hazard rate $h(x)$ eventually decreasing to 0. Then $F \in S$ if and only if

$$\lim_{x \to \infty} \int_{0}^{x} \exp \{yh(x)\} f(y) \, dy = 1$$

(17)
Theorem

Suppose $F$ is absolutely continuous with density function $f$ and hazard rate $h(x)$ with $\alpha(h) < 0$. Then $F \in S$ if and only if

$$\lim_{x \to \infty} \int_{0}^{x} \exp \{ kyh(x) \} f(y) \, dy = 1$$

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for all $k > 0$. 

Suppose $F$ is absolutely continuous with density function $f$ and hazard rate $h(x)$ with $\alpha(h) < 0$. Then $F \in S$ if and only if

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BIBLIOGRAPHY


Thank you for your attention