# Characterization through Hazard Rate of heavy tailed distributions and some Convolution Closure Properties

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## INTRODUCTION

We will assume for the whole paper that the distribution function $F$ is supported on $[0, \infty)$ and that $F$ has positive Lebesque density function $f$.
We use the Matuszweska indices and its properties to show asymptotics inequalities for the hazard rates. We discuss about the relation membership in dominatedly or subversively varying tail distribution and a hazard rate condition. Convolution closure is establish for the class of distributions with subexponential and subversiverly varying tails.

## The class $\mathcal{L}$

A distribution function F belongs to the class $\mathcal{L}$ if

$$\lim_{x \to \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1$$

for all constants $y \in R$.
This distribution function F is said to have a long tail.

## The class $\mathcal{D}$

A distribution function F belongs to the class $\mathcal{D}$ if

$$\limsup_{x\to\infty} \frac{\overline{F}(ux)}{\overline{F}(x)} < \infty \tag{1}$$

for any $0 < u < 1$ (or equivalently for $u = 1/2$). Such a distribution function F is said to have a dominatedly varying tail. An equivelant way to right (1) is

$$\overline{F}_*(u) := \liminf_{x\to\infty} \frac{\overline{F}(ux)}{\overline{F}(x)} > 0 \tag{2}$$

for any $u > 1$.

## The class $\mathcal{E}$

A distribution function F belongs to the class $\mathcal{E}$, if for some $u > 1$

$$\overline{F}^{*}(u) := \limsup_{x\to\infty} \frac{\overline{F}(ux)}{\overline{F}(x)} < 1.$$

Such a distribution function F is said to have subversively varying tail.

## The class $\mathcal{S}$

A distribution function F belongs to the class $\mathcal{S}$ if

$$\lim_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n$$

for any $n \geq 2$ (or equivalently for $n = 2$), where $F^{*n}$ denotes the $n$th convolution of F. Such a distribution function F is said to have a subexponetial tail.

## The class $\mathcal{A}$

A distribution function F belongs to the class $\mathcal{A}$ if

$$F \in \mathcal{S} \text{ and } \overline{F}^{*}(u) < 1\,,$$

for some $u > 1$. In others words $\mathcal{A} = \mathcal{E} \cap \mathcal{S}$.

It is well known the following inclusions

$$\mathcal{D} \cap \mathcal{A} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}.$$

$$\mathcal{D} \cap \mathcal{A} \subset \mathcal{D} \cap \mathcal{E}$$

## Definition of Matuszewska indices

Upper Matuszewska index

Let g be positive. Then
The upper Matuszewska index $\alpha(g)$ is the infimum of those $\alpha$ for which there exists a constant $C = C(\alpha) > 0$ such that for each $\Lambda > 1$

$$\frac{g(\lambda x)}{g(x)} \leq C(1 + o(1))\lambda^{\alpha},$$

as $x \to \infty$ and uniformly in $\lambda \in [1, \Lambda]$.

## Lower Matuszewska index

Let g be positive. Then
The lower Matuszewska index $\beta(g)$ is the supremum of those $\beta$ for which there exists a constant $D = D(\beta) > 0$ such that for each $\Lambda > 1$

$$\frac{g(\lambda x)}{g(x)} \geq D(1 + o(1))\lambda^{\beta},$$

as $x \to \infty$, and uniformly in $\lambda \in [1, \Lambda]$.

Let consider the following notation. The positive function $g$ has

1. bounded increase, ($g \in BI$), if $\alpha(g) < \infty$
2. bounded decrease, ($g \in BD$), if $\beta(g) > -\infty$
3. positive increase, ($g \in PI$), if $\beta(g) > 0$
4. positive decrease, ($g \in PD$), if $\alpha(g) < 0$

An equation that holds between the Matusweska indices is

$$\beta(g) = -\alpha\left(\frac{1}{g}\right), \quad \textit{for } g \textit{ positive} \tag{3}$$

For more details of the Matuszewska indices, see Chapter 2.1 of Bingham *et al.* (1987).

## Potter Inequalities

Let g(.) be positive.
If g $\in$ BI then for every $\alpha > \alpha(g)$ there exist positive constants $C$ , $x_0$ such that

$$\frac{g(y)}{g(x)} \leq C \left(\frac{y}{x}\right)^{\alpha}, \quad (y \geq x \geq x_0) \tag{4}$$

If g $\in$ BD then for every $\beta < \beta(g)$ there exist positive constants $C^{'}$ , $x_0$ such that

$$\frac{g(y)}{g(x)} \geq C^{'} \left(\frac{y}{x}\right)^{\beta}, \quad (y \geq x \geq x_0) \tag{5}$$

## Matuszewska indices for the distributions tails

From D. B. H. Cline and G. Samorodnisky(1994)
The upper Matuszewska index $\gamma_{\overline{F}}$ for a distribution function F, was introduced as follows.

$$\gamma_{\overline{F}} := \inf\left\{-\frac{\log \overline{F}_*(u)}{\log u} : u > 1\right\} = -\lim_{u\to\infty} \frac{\log \overline{F}_*(u)}{\log u}. \qquad (6)$$

The lower Matuszewska index $\delta_{\overline{F}}$ for a distribution function F, was introduced as follows.

$$\delta_{\overline{F}} := \sup\left\{-\frac{\log \overline{F}^*(u)}{\log u} : u > 1\right\} = -\lim_{u\to\infty} \frac{\log \overline{F}^*(u)}{\log u}. \qquad (7)$$

## Inequalities for the distributions functions

If the Upper Matuszeska index $\gamma_{\overline{F}} < \infty$ then there exist constants $C, x_0$ such that

$$\frac{\overline{F}(x)}{\overline{F}(y)} \leq C \left(\frac{x}{y}\right)^{-\gamma} \tag{8}$$

for every $x \geq y \geq x_0$ and $\gamma_{\overline{F}} < \gamma < \infty$.
If the Lower Matuszeska index $\delta_{\overline{F}} > 0$ is finite then there exist constants $C^{'}, x_0$ such that

$$\frac{\overline{F}(x)}{\overline{F}(y)} \geq C^{'} \left(\frac{x}{y}\right)^{-\delta} \tag{9}$$

for every $x \geq y \geq x_0$ and $0 < \delta < \delta_{\overline{F}}$.

## Matuszewska indices for the density functions

We introduced the upper Matuszewska index $\gamma_f$ for the density function, as follows

$$\gamma_f := \inf \left\{ -\frac{\log f_*(u)}{\log u} : u > 1 \right\} = -\lim_{u \to \infty} \frac{\log f_*(u)}{\log u}. \qquad (10)$$

where $f_*(u) = \liminf_{x \to \infty} \dfrac{f(ux)}{f(x)}$.

We introduced the lower Matuszewska index $\delta_f$ for the density function, as follows

$$\delta_f := \sup \left\{ -\frac{\log f^*(u)}{\log u} : u > 1 \right\} = -\lim_{u \to \infty} \frac{\log f^*(u)}{\log u}. \qquad (11)$$

where $f_*(u) = \limsup_{x \to \infty} \dfrac{f(ux)}{f(x)}$.

## Definition

We will say that the density function has

1. bounded decrease, $(f \in BD)$ , if $\gamma_f = \alpha(f^{-1}) < \infty$
2. positive decrease, $( f \in PD)$, if $\delta_f = \beta(f^{-1}) > 0$

In analogy to inequalities (8) and (9) we introduce inequalities for density function

## Corollary

*If $f \in BD$ then there exist constants $C, x_0$ such that*

$$\frac{f(y)}{f(x)} \geq C' \left(\frac{y}{x}\right)^{-\gamma} \tag{12}$$

*for every $y \geq x \geq x_0$.*
*If $f \in PD$ then there exist constants $C', x_0$ such that*

$$\frac{f(y)}{f(x)} \leq C \left(\frac{y}{x}\right)^{-\delta} \tag{13}$$

*for every $y \geq x \geq x_0$.*

## Subversively Varying Tail

The Subversively Class is a large class that extends out of the class of Heavy Tail. We can see the following Example

**Example**

The Exponential distribution function is $\overline{F}(x) = \exp\{-\lambda x\}$. As we can see

$$\lim_{x \to \infty} \frac{\overline{F}(ux)}{\overline{F}(x)} = 0 < 1$$

for all $u > 1$.

## Lemma (Konstantinides et.al 2002)

*Let F be a d.f. with a density function f which is eventually non-increasing. Then the following statements are equivelant:*

1. $\overline{F}^*(u) < 1$ *holds for some* $u > 1$;
2. $\overline{F}^*(u) < 1$ *holds for any* $u > 1$;
3. *the hazard rate function of* $F, h(x) = \dfrac{f(x)}{\overline{F}(x)}$ *satisfies*

$$\liminf_{x \to \infty} xh(x) > 0.$$

## Eventually non increasing?

If a function g is eventually non increasing then

$$\forall y \geq x \geq x_0 \;\Rightarrow\; g(y) \leq g(x) \tag{14}$$

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- Does the condition (14) hold for every function g?

We can obtain that condition (14) is the Potter inequality (4) for $C \leq 1$ and $\alpha(g) < 0$. Obviously

$$\frac{g(y)}{g(x)} \leq C \left(\frac{y}{x}\right)^{\alpha} \leq C\,, \quad (x \geq y \geq x_0). \tag{15}$$

for all $\alpha(g) < \alpha < 0$. But to obtain, if the Potter inequality holds we only need to calculate the Upper Matuswzeska index

The main idea is to replace the condition of eventually non-increasing(or decreasing) with the Potter Inequalities. When we have density functions we can obtain an analogous inequality of the previous one, if the $\delta_f > 0$.

**Theorem**

*Let $F$ be an absolute continuous distribution function supported on $[0, \infty)$ with a density function $f(x)$ with $\delta_f > 0$ $(f \in PD)$. Then*

$$F \in \mathcal{E} \text{ if and only if } \liminf_{x \to \infty} xh(x) > 0. \tag{16}$$

A sufficient condition for $F \in \mathcal{E}$ is given from the following theorem

**Theorem**

*If* $\liminf_{x\to\infty} xh(x) > 0$ *then* $F \in \mathcal{E}$

A similar theorem is found in Klüppelberg C.(1988) for the class $\mathcal{D} \cap \mathcal{L}$.

## Lemma

*If* $f \in PD$ , $\delta_f > 1$ *for any* $\delta \in (1\,,\delta_f)$ *then :*

$$xh(x) = x\frac{f(x)}{\overline{F}(x)} \geq \frac{(\delta - 1)}{C} > 0$$

*for all* $x \geq x_0$ *and* $C > 0$.

From Lemma and previous theorem we obtain

**Lemma**

*If* $f \in PD\,, \delta_f > 1$ *for any* $\delta \in (1\,, \delta_f)$ *then :*

$$xh(x) = x\frac{f(x)}{\overline{F}(x)} \geq \frac{(\delta - 1)}{C} > 0$$

*for all* $x \geq x_0$ *and* $C > 0$.

From Lemma and previous theorem we obtain

**Corollary**

*If* $f \in PD$ *with* $\delta_f > 1$ *then* $F \in \mathcal{E}$.

**Theorem**

*If $F_1 \, , F_2 \in \mathcal{E}$ and the following statements hold*

1. $f_1 \in PD$
2. $x^{-\delta^*} = O\left(\frac{1}{\overline{F_1}(x))}\right)$

*where* $\delta^* = \min\left(\delta_{\overline{F}_1} \, , \delta_{\overline{F}_2}\right)$, *then* $F_1 * F_2 \in \mathcal{E}$.

## Subexponential Class

A sufficient condition for $F \in \mathcal{D} \cap \mathcal{L}$ is given by the following theorem.

**Theorem (Klüppelberg C. 1988)**

. *If* $\limsup_{x\to\infty} xh(x) < \infty$ *then* $F \in \mathcal{D} \cap \mathcal{L}$

**Corollary (Klüppelberg C. 1988)**

*Let F have an eventually decreasing density f. Then the following statements are equivelant:*

1. $F \in \mathcal{D}$
2. $F \in \mathcal{D} \cap \mathcal{L}$
3. $\limsup_{x\to\infty} xh(x) < \infty$

**Theorem**

*Let F be an absolute continuous distribution function supported on* $[0, \infty)$ *with a density function* $f(x)$ *with* $\delta_f > 0$ $(f \in PD)$. *Then*

1. $F \in \mathcal{D}$
2. $F \in \mathcal{D} \cap \mathcal{L}$
3. $\limsup_{x\to\infty} xh(x) < \infty$

**Lemma**

*If $f \in BD$ then there is positive $x_0$, such that for all $x \geq x_0$ and all $\lambda > 1$:*

$$xh(x) = x\frac{f(x)}{\overline{F}(x)} \leq \frac{-\gamma + 1}{C'(\lambda^{-\gamma+1} - 1)}$$

*Furthermore if $f \in BD$ then $F \in \mathcal{D} \cap \mathcal{L}$.*

## The class $\mathcal{A} \cap \mathcal{D}$

**Theorem**

*Let F be an absolute continuous distribution function with a density function $f(x)$ with $\delta_f > 0$ then $F \in \mathcal{A} \cap \mathcal{D}$ if and only if one of the following statements holds*

1. $0 < \liminf_{x\to\infty} xh(x) \leq \limsup_{x\to\infty} xh(x) < \infty$
2. $0 < \overline{F}_\star(u) \leq \overline{F}^\star(u) < 1$

## Pitman (A Characterization Theorem for $\mathcal{S}$ )

**Theorem (Pitman(1980))**

*Suppose F is absolutely continuous with density function f and hazard rate $h(x)$ eventually decreasing to 0. Then $F \in \mathcal{S}$ if and only if*

$$\lim_{x\to\infty} \int_0^x \exp\{yh(x)\} f(y)dy = 1 \tag{17}$$

**Theorem**

*Suppose F is absolutely continuous with density function f and hazard rate* $h(x)$ *with* $\alpha(h) < 0$. *Then* $F \in \mathcal{S}$ *if and only if*

$$\lim_{x \to \infty} \int_0^x \exp\{kyh(x)\} f(y) dy = 1 \tag{18}$$

*for all* $k > 0$.

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## Thank you for your attention