Ratio monotonicity for tail probabilities in the renewal risk model

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Renewal risk model

The surplus of the insurer at time t is of the form

$$U(t) = u + ct - \sum_{k=1}^{N_t} Y_k, \tag{1}$$

where

- $u = U(0)$: insurer's initial surplus,
- c : rate of premium income per unit time,
- N_t : number of claims in the time interval (0,t],
- $T_1, T_2, \ldots$: interarrival times among successive claims,
- $Y_1, Y_2, \ldots$: individual claim amounts with distribution P in $(0, \infty)$ and mean value $m < \infty$.

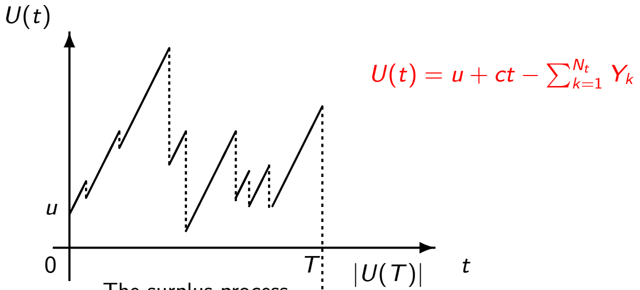
Ruin probabily

We assume that $m < c\,E(T_1)$, so that ruin is not certain to occur. The probability of ruin is then defined by

$$\psi(u) = Pr\left\{\inf_{t>0} U(t) < 0 \,|\, U(0) = u\right\}.$$

If T is the time of ruin, then

$$\psi(u) = Pr(T < \infty \,|\, U(0) = u).$$



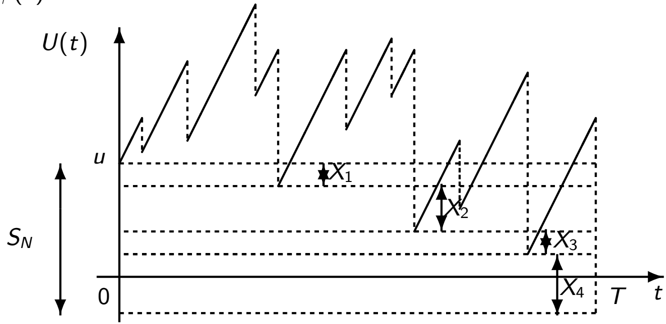
The surplus process.

Ladder hights

In the figure, the r.v. $X_1, X_2, X_3, \ldots$ are i.i.d. with d.f. F, and they are called ladder heights. Moreover,

$$\psi(u) = Pr(S_N > u) = \sum_{n=1}^{\infty}(1-\phi)\phi^n \overline{F^{*n}}(u),$$

where $\phi = \psi(0)$.



The ladder heights.

Tail of the deficit

The tail of deficit at ruin is

$$\overline{G}(u,y) = Pr(|U(T)| > y, T < \infty | U(0) = u),$$

and we note that $\overline{G}(u,0) = \overline{G}(u) = \psi(u) = Pr(T < \infty \,|\, U(0) = u) < 1$.

By Willmot (2002), we know that $\overline{G}(u,y)$ satisfies the d.r.e.

$$\overline{G}(u,y) = \phi \int_0^u \overline{G}(u-t,y)\, dF(t) + \phi \overline{F}(u+y)$$

whose solution is

$$\overline{G}(u,y) = \frac{\phi}{1-\phi} \int_{0+}^u \overline{F}(u+y-t)\, dG(t) + \phi \overline{F}(u+y),$$

where $G(u) = 1 - \psi(u) = \sum_{n=0}^{\infty}(1-\phi)\phi^n F^{*n}(u)$, with $\phi = \psi(0)$.

Monotonicity of the ratio $\overline{G}(u,y)/\overline{F}(u+y)$

Recall that the individual claim amounts $Y_1, Y_2, \cdots \overset{i.i.d.}{\sim} P$ with mean value m.

Embrechts and Veraverbeke (1982) proved that

$$\lim_{u\to\infty} \frac{\overline{P_e^{*n}}(u)}{\overline{P_e}(u)} = n \Rightarrow \lim_{u\to\infty} \frac{\psi(u)}{\overline{P_e}(u)} = \frac{\phi}{1-\phi},$$

where $\overline{P_e}(u) = 1 - P_e(u) = \int_u^\infty \overline{P}(t)\,dt/m$ is the tail of the equilibrium d.f. P_e of P.

In the classical risk model, where the number of claims N_t is a Poisson process at rate λ, the ladder height d.f., F, satisfies $F = P_e$.

Recently, Psarrakos (2009) generalized the above result to

$$\lim_{u\to\infty} \frac{\overline{G}(u,y)}{\overline{P_e}(u+y)} = \frac{\phi}{1-\phi},$$

for any $y \geq 0$. Recall that $\overline{G}(u,0) = \psi(u)$.

Monotonicity of $\overline{G}(u,y)/\overline{F}(u+y)$ for DFR and stable distributions

$X \sim F$ is DFR (IFR) if

$$Pr(X - x > y | X > x) = \frac{\overline{F}(x+y)}{\overline{F}(x)} \uparrow (\downarrow) \text{ in } x.$$

$X \sim F$ is *stable* if for any $A, B > 0$, there exists $C > 0$ and $D \in \mathbb{R}$ such that

$$A\,X_1 + B\,X_2 \overset{d}{=} C\,X + D,$$

where X_1 and X_2 are independent copies of X. Moreover, there is an *index* a of a stable r.v. such that $C^a = A^a + B^a$, $a \in (0, 2]$. Stable distributions on $[0, \infty)$ with index $a \in (0, 2)$ form a subclass of the subexponential distributions, i.e.,

$$\lim_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n, \quad n \geq 2.$$

Lemma (Daley et al., 2007)
If F is stable, then the ratio $\overline{F^{*n}}(x)/\overline{F}(x)$ is nondecreasing in x, for any $n \geq 2$.

We obtain an upper bound for $\overline{F^{*n}}(x)/\overline{F}(x), \quad n = 2, 3, \ldots$ as a function of $\overline{F^{*2}}(x)/\overline{F}(x)$.

Proposition
If F is stable, then for any $x \geq 0$ and any $n \geq 2$,

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq \sum_{k=0}^{n-3} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^k + \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 \right]^{n-2}. \qquad (2)$$

(By convention, $\sum_{k=0}^{s} [\overline{F^{*2}}(x)/\overline{F}(x) - 1]^k = 0$ for any $s < 0$.)

Recall that $\psi(u) = Pr(S_N > u) = \sum_{n=1}^{\infty}(1-\phi)\phi^n\overline{F^{*n}}(u)$, where $\phi = \psi(0)$.

Lemma (Daley et al., 2007)
If F is stable, then the ratio $\overline{F^{*n}}(x)/\overline{F}(x)$ is nondecreasing in x, for any $n \geq 2$.

Lemma
If F is stable, then $\psi(u)/\overline{F}(u)$ is nondecreasing in u.

Lemma (Psarrakos and Politis (2009))
If F is DFR, then $\overline{G}(u,y)/\psi(u+y)$ is nondecreasing in u.

Theorem
If F is DFR and stable, then $\overline{G}(u,y)/\overline{F}(u+y)$ is nondecreasing in u and

$$\lim_{u\to\infty}\frac{\overline{G}(u,y)}{\overline{F}(u+y)} = \frac{\phi}{1-\phi}.$$

Monotonicity of $\overline{G}(u,y)/\overline{F}(u+y)$ for phase-type distributions

For simplicity, we consider the classical model of risk theory with Poisson arrivals and phase-type claim size distribution.

- By $\mathbf{1}$ we denote a column vector of all ones.
- When
$$|a_{ii}| \geq \sum_{i \neq j} |a_{ij}|, \quad i = 1, 2, \ldots, m, \tag{3}$$
we refer to $A = [a_{ij}] \in \mathbb{R}^{m \times m}$ as a *diagonally dominant* matrix.
- We call $A = [a_{ij}] \in \mathbb{R}^{m \times m}$ a *subintensity matrix* if $a_{ij} \geq 0$ for all $i \neq j$, $a_{ii} \leq 0$ for $i = 1, 2, \ldots, m$ and A is diagonally dominant with at least one inequality in (3) being strict.
- The matrix exponential e^{xA}, and its derivative relative to $x \in \mathbb{R}$ are
$$e^{xA} = \sum_{k=0}^{\infty} \frac{x^k A^k}{k!} \quad \text{and} \quad \frac{d\, e^{xA}}{dx} = \sum_{k=1}^{\infty} \frac{x^{k-1} A^k}{(k-1)!} = Ae^{xA}.$$

Let A be the $m \times m$ transition matrix of a *continuous time Markov chain* with a single absorbing state 0 and m transient states. The row vector β^T contains the probabilities b_j that the process starts in the various transient states $j = 1, 2, \ldots, m$. We write $P = PH_m(\beta, A)$, i.e., the claim size has a phase-type distribution with representation (β, A) of dimension m, having d.f.

$$P(x) = 1 - \beta^T\, e^{xA}\, \mathbf{1}, \quad x \geq 0.$$

In what follows, we assume that A is an invertible subintensity matrix.
The ladder height d.f., the maximal aggregate loss (ruin probability) and the deficit at ruin are also phase-type distributions and their tails are given by

$$\overline{F}(u) = \hat{\beta}^T\, e^{Au}\, \mathbf{1},$$

$$\psi(u) = \phi\, \hat{\beta}^T\, e^{u(A + \phi\, \alpha\, \hat{\beta}^T)}\, \mathbf{1}$$

$$\overline{G}(u, y) = \phi\, \hat{\beta}^T\, e^{Ay}\, e^{u(A + \phi\, \alpha\, \hat{\beta}^T)}\, \mathbf{1}$$

respectively, where $\hat{\beta}^T = -\mu^{-1} \beta^T A^{-1}$ and $\alpha = -A\mathbf{1}$.

Next we derive the monotonicity, as a function of u, of the ratio

$$\frac{\overline{G}(u,y)}{\overline{F}(u+y)} = \frac{\phi\,\hat{\beta}^T\, e^{yA}\, e^{u(A+\phi\alpha\hat{\beta}^T)}\,\mathbf{1}}{\hat{\beta}^T\, e^{yA}\, e^{uA}\,\mathbf{1}}. \tag{4}$$

Theorem

If the claim size $P = PH_m(\beta, A)$ has a phase-type distribution with representation (β, A), then the ratio $\overline{G}(u,y)/\overline{F}(u+y)$ in (4) is a nondecreasing function of $u \in [0,\infty)$.

Steps of the proof: It is enough to establish the following

(1) $\hat{\beta} \geq 0$,

(2) $e^{yA}\, e^{uA} \geq 0$ and $e^{yA}\, e^{u(A+\phi\alpha\hat{\beta}^T)} \geq 0$ for all $u \geq 0$, and

(3) $\dfrac{d\, e^{yA}\, e^{u(A+\phi\alpha\hat{\beta}^T)}}{du} \geq \dfrac{d\, e^{yA}\, e^{uA}}{du}$.

Example 1

Suppose the claim amount d.f. is the mixture of exponentials with density

$$p(x) = \frac{1}{2}\, e^{-x} + e^{-2x}, \quad x \geq 0,$$

and assume that the relative safety loading is $\theta = 0.6$ ($\phi = \psi(0) = 1/(1+\theta)$). Such a distribution is phase type and DFR. The ladder height d.f. F is also a mixture of exponentials with tail

$$\overline{F}(x) = \frac{\int_x^\infty \overline{P}(t)\,dt}{\int_0^\infty \overline{P}(t)\,dt} = \frac{2}{3}\, e^{-x} + \frac{1}{3}\, e^{-2x},$$

the probability of ruin is given by

$$\psi(u) = 0.592581\, e^{-0.432479} + 0.032419\, e^{-1.73419}$$

and the tail of the d.f. of the deficit at ruin is

$$\begin{aligned}\overline{G}(u,y) &= 0.117504\, e^{-1.73419u-2y} + 0.0908294\, e^{-0.432479u-2y} \\ &\quad -0.0850836\, e^{-1.73419u-y} + 0.501751\, e^{-0.4324794-y}.\end{aligned}$$

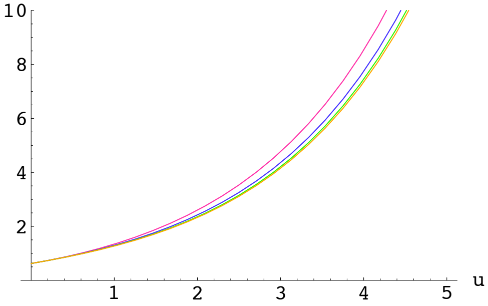


Figure: Monotonicity for $\overline{G}(u, y)/\overline{F}(u + y)$ in u for $y = 0, 1, 2, 3$.

We observe in Figure 1 that the ratio $\overline{G}(u, y)/\overline{F}(u + y)$ is nondecreasing in u.

$$\frac{\overline{G}(u, y)}{\overline{F}(u + y)} = \begin{cases} \color{red}{\rule{3em}{1pt}} & \text{for y} = 0 \\ \color{blue}{\rule{3em}{1pt}} & \text{for y} = 1 \\ \color{green}{\rule{3em}{1pt}} & \text{for y} = 2 \\ \color{yellow}{\rule{3em}{1pt}} & \text{for y} = 3 \end{cases}$$

Moreover, by Psarrakos and Politis (2009), since P is DFR, $\overline{G}(u,y)/\overline{F}(u+y)$ is nonincreasing in y,

$$\text{——} \geq \text{——} \geq \text{——} \geq \text{——},$$

namely,

$$\frac{\psi(u)}{\overline{F}(u)} \geq \frac{\overline{G}(u,1)}{\overline{F}(u+1)} \geq \frac{\overline{G}(u,2)}{\overline{F}(u+2)} \geq \frac{\overline{G}(u,3)}{\overline{F}(u+3)}.$$

Example 2

Let the interclaims arrivals have an Erlang $(2,2)$ d.f. with density $k(x) = 4xe^{-2x}$. Also assume that the claim amount d.f. is Erlang $(2,2)$ with density $p(x) = 4xe^{-2x}$.
For $c = 1.1$, by Dickson (1998), it follows that

$$\overline{F}(u) = e^{-2y} + 0.8217e^{-2y}\,y,$$

$$\psi(u) = 0.9283e^{-0.1818u} - 0.0192e^{-2.7892u}$$

and

$$\begin{aligned}\overline{G}(u,y) &= 0.8841e^{-0.1818u-2y} - 0.0109e^{-2.7892u-2y} \\ &\quad +0.5003e^{-0.1818u-2y}\,y + 0.2172e^{-2.7892u-2y}\,y.\end{aligned}$$

Erlang $(2,2)$ is a phase type distribution.

Observe that the ratio $\overline{G}(u, y)/\overline{F}(u + y)$ is nondecreasing in u and since F is IFR,

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$\overline{G}(u, y)/\overline{F}(u + y)$ is nondecreasing in y (see Psarrakos and Politis (2009)).

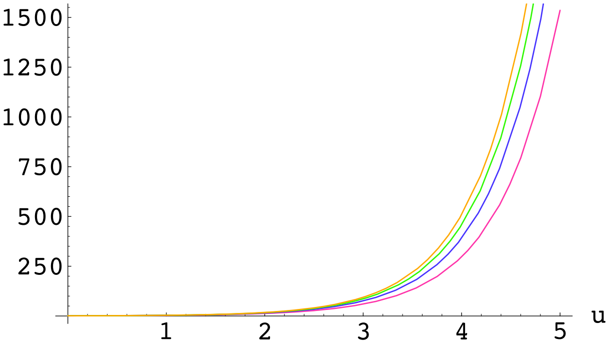

Figure: Monotonicity for $\overline{G}(u, y)/\overline{F}(u + y)$ in u for $y = 0, 1, 2, 3$.

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