The Finite-time Ruin Probability with Heavy-tails and Dependent Return Rates [1]

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Outline

Model Descriptions

- Within period i the insurer's total premium income is denoted by a nonnegative random variable A_i, and total claim amount plus other daily costs is denoted by another nonnegative random variable B_i. We assume that (A_i, B_i) are i.i.d. copies of a random vector (A, B) whose components are however dependent.
- Suppose that the insurer positions him/herself in a discrete-time financial market consisting of a risk-free bond with a constant periodic interest rate $r > 0$ and a risky stock with a random periodic return rate $R_i \in (-1, \infty)$ during period i. We assume that R_i are dependent on each other according to a suitably chosen multivariate distribution but they are independent of (A_i, B_i).

- Suppose that, in the beginning of each period i, the insurer invests a fraction $\pi_i \in [0, 1]$ of the current wealth in the stock and keeps the remaining wealth in the bond.

- Denote by U_i the insurer's wealth at time i, with a deterministic initial value $U_0 = x \geq 0$. Then, we have the recursive equation

$$U_i = [(1 - \pi_i)(1 + r) + \pi_i(1 + R_i)]\, U_{i-1} + (A_i - B_i)\,. \quad (1)$$

Insurance Risk and Financial Risk

For simplicity, introduce

$$Q_i = 1 + R_i, \quad X_i = B_i - A_i, \quad Y_i = \frac{1}{(1-\pi_i)(1+r) + \pi_i Q_i}. \quad (2)$$

- Q_i: the accumulation factor during period i of the risky asset and it takes values in $(0, \infty)$
- X_i: the net loss during period i
- Y_i: overall scholastic discount rate from time i to time $i-1$

We call X_i the insurance risks and Y_i the financial risks. According to the assumptions above, these insurance risks are i.i.d. with common distribution F, say, and are independent of the financial risks.

Finite-Time Ruin Probability

Define the probability of ruin by time n as

$$\psi(x;n) = P\left(\min_{0\le i\le n} U_i < 0 \,\middle|\, U_0 = x \right).$$

Iterating (1) and multiplying both sides by $\prod_{j=1}^{i} Y_j$, we obtain the stochastic present values of U_i as

$$\widetilde{U}_0 = x, \quad \widetilde{U}_k = x - \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j = x - V_k.$$

In this way,

$$\psi(x;n) = P\left(\max_{1\le k\le n} V_k > x \right). \tag{3}$$

Definition

We assume that the net loss variable X follows a subexponential distribution F.

By definition, a distribution F on $[0,\infty)$ is said to be subexponential, denoted as $F \in \mathcal{S}$, if the relation

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2$$

holds, where F^{2*} denotes the 2-fold convolution of F. See Embrechts-Klüppelberg-Mikosch (1997) for a nice review on subexponential distributions.

More generally, a distribution F on $(-\infty,\infty)$ is still said to be subexponential if $F_+(x) = F(x)1_{(x\geq 0)}$ is.

Principle of a Single Big Jump

A remarkable feature of subexponentiality is that, for i.i.d. random variables X_i with common distribution $F \in \mathcal{S}$, the relation

$$\lim_{x\to\infty} \frac{P\left(\sum_{i=1}^{n} X_i > x\right)}{P\left(\max_{1\le i\le n} X_i > x\right)} = 1.$$

The principle of a single big jump says that an extreme event happens mainly due to a single, unusually large, input to the stochastic system. Thus, the concept of subexponentiality builds the theoretical basis for the principle of a single big jump.

The assumption is relevant especially nowadays, in view of the September 11, 2001 attacks, the 2004 Indian Ocean Tsunami, the 2005 Hurricane Katrina, the 2008 Sichuan earthquake, the 2010 Haiti earthquake, and, in particular, the recent financial tsunami.

Theorem 1 - Main Result

Our primary goal in this study is to derive some exact asymptotic formulas for the ruin probability $\psi(x; n)$ with the requirement that the obtained formulas capture the impacts of the underlying dependence structures.

Theorem 1. If $X_1, \ldots, X_n$ are i.i.d. with common distribution $F \in \mathcal{S}$, then

$$\psi(x; n) \sim \sum_{i=1}^{n} P\left(X_i \prod_{j=1}^{i} Y_j > x\right). \tag{4}$$

This is an easy consequence of the following result.

Theorem 2 - An Important Lemma

Let $X_1, \ldots, X_n$ be n i.i.d. real-valued random variables with common distribution F, and let $\theta_1, \ldots, \theta_n$ be other n positive random variables independent of $X_1, \ldots, X_n$.

Theorem 2. If $F \in \mathcal{S}$ and the random variables $\theta_1, \ldots, \theta_n$ are positive and bounded from above, then

$$P\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \sim \sum_{i=1}^{n} P\left(\theta_i X_i > x\right).$$

An important feature of this result is that it does not require any specific information about the dependence structure of $\theta_1, \ldots, \theta_n$.

Sketch of the proof of Theorem 2

For arbitrarily given small $\varepsilon > 0$, we rewrite

$$\begin{aligned} & P\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \\ = \; & P\left(\sum_{i=1}^{n} \theta_i X_i > x, \bigcap_{i=1}^{n} (\theta_i > \varepsilon)\right) + P\left(\sum_{i=1}^{n} \theta_i X_i > x, \bigcup_{i=1}^{n} (\theta_i \leq \varepsilon)\right) \\ = \; & I_1(x, \varepsilon) + I_2(x, \varepsilon). \end{aligned}$$

Note that each θ_i in $I_1(x,\varepsilon)$ is two-sided bounded. Hence, we may apply Theorem 3.1 of Tang-Tsitsiashvili (2003) to get

$$\begin{aligned} I_1(x,\varepsilon) &= P\left(\sum_{i=1}^{n}\theta_i X_i > x \,\middle|\, \bigcap_{i=1}^{n}(\theta_i > \varepsilon)\right) P\left(\bigcap_{i=1}^{n}(\theta_i > \varepsilon)\right) \\ &\sim \sum_{i=1}^{n} P\left(\theta_i X_i > x \,\middle|\, \bigcap_{i=1}^{n}(\theta_i > \varepsilon)\right) P\left(\bigcap_{i=1}^{n}(\theta_i > \varepsilon)\right) \\ &= \sum_{i=1}^{n} P\left(\theta_i X_i > x\right) - \sum_{i=1}^{n} P\left(\theta_i X_i > x, \bigcup_{i=1}^{n}(\theta_i \le \varepsilon)\right) \\ &= \sum_{i=1}^{n} P\left(\theta_i X_i > x\right) - I_3(x,\varepsilon). \end{aligned}$$

It remains to prove that

$$\lim_{\varepsilon\to 0}\limsup_{x\to\infty}\left|\frac{I_2(x,\varepsilon)-I_3(x,\varepsilon)}{\sum_{i=1}^{n}P\left(\theta_i X_i>x\right)}\right|=0.$$

As an illustration, we only consider $n=2$ and the following term:

$$\begin{aligned}J(x,\varepsilon) &= P\left(\theta_1X_1+\theta_2X_2>x,\theta_1\le\varepsilon,\theta_2>\varepsilon\right)\\ &\quad -P\left(\theta_1X_1>x,\theta_1\le\varepsilon,\theta_2>\varepsilon\right)\\ &\quad -P\left(\theta_2X_2>x,\theta_1\le\varepsilon,\theta_2>\varepsilon\right).\end{aligned}$$

By Theorem 3.1 of Tang-Tsitsiashvili (2003) again,

$$\begin{aligned}&P\left(\theta_1X_1+\theta_2X_2>x,\theta_1\le\varepsilon,\theta_2>\varepsilon\right)\\ \le\ &P\left(\varepsilon X_1+\theta_2X_2>x,\theta_1\le\varepsilon,\theta_2>\varepsilon\right)\\ \sim\ &P\left(\varepsilon X_1>x,\theta_1\le\varepsilon,\theta_2>\varepsilon\right)+P\left(\theta_2X_2>x,\theta_1\le\varepsilon,\theta_2>\varepsilon\right).\end{aligned}$$

Thus,

$$
\begin{aligned}
& \left| \frac{J(x,\varepsilon)}{\sum_{i=1}^{2} P\left(\theta_i X_i > x\right)} \right| \\
\lesssim \quad & \left| \frac{P\left(\varepsilon X_1 > x, \theta_1 \le \varepsilon, \theta_2 > \varepsilon\right) - P\left(\theta_1 X_1 > x, \theta_1 \le \varepsilon, \theta_2 > \varepsilon\right)}{\sum_{i=1}^{2} P\left(\theta_i X_i > x\right)} \right| \\
\le \quad & 2 \frac{P\left(\varepsilon X_1 > x, \theta_1 \le \varepsilon, \theta_2 > \varepsilon\right)}{P\left(\theta_1 X_1 > x\right)} \\
= \quad & 2 \frac{P\left(\varepsilon X_1 > x, \theta_1 > \varepsilon, \theta_2 > \varepsilon\right)}{P\left(\theta_1 X_1 > x\right)} \frac{P\left(\theta_1 \le \varepsilon, \theta_2 > \varepsilon\right)}{P\left(\theta_1 > \varepsilon, \theta_2 > \varepsilon\right)}.
\end{aligned}
$$

Notice that the first fraction is bounded by 1 and the second fraction tends to 0 as $\varepsilon \to 0$.

Some Questions on Theorem 1

Question 1. Under what conditions on (A, B) can $X = B - A$ be subexponentially distributed.

Our ideas: If A and B are independent, then X is subexponentially distributed if and only if B is. For dependent cases, in addition to the subexponentiality of B, we need to add some mild conditions on the dependence structure of (A, B), say, through a copula.

Question 2. Are the tail probabilities $P\left(X_i \prod_{j=1}^{i} Y_j > x\right)$ appearing in (4) computable?

Our Ideas: More explicit expressions can be given if further assuming that the distribution of X also belongs to the maximum domain of attraction of an extreme value distribution.

Maximal Domains of Attraction

A distribution function F belongs to the max-domain of attraction of an extreme value distribution H, written as $F \in \mathrm{MDA}(H)$, if

$$\lim_{n\to\infty} \sup_{-\infty<x<\infty} |F^n(a_n x + b_n) - H(x)| = 0 \tag{5}$$

holds for some positive a_n and real-valued b_n, $n \geq 1$.

Only three choices for H are possible, namely the Fréchet distribution, the Gumbel distribution, or the Weibull distribution.

Write r_F as the upper endpoint of the distribution F.

The Fréchet Case

The functional form of the Fréchet distribution is $\Phi_\gamma(x) = \exp(-x^{-\gamma})$ for $\gamma > 0$ and $x > 0$. If $F \in \mathrm{MDA}(\Phi_\gamma)$, then $r_F = \infty$ and (5) with $H = \Phi_\gamma$ is equivalent to

$$\lim_{x\to\infty} \frac{\overline{F}(xu)}{\overline{F}(x)} = u^{-\gamma}, \quad \forall u > 0.$$

This means that $\overline{F}$ is regularly varying at infinity with index $-\gamma$.

The Gumbel Case

The Gumbel distribution is given by $\Lambda(x) = \exp(-\exp(-x))$ for all real x. If $F \in \mathrm{MDA}(\Lambda)$, then (5) with $H = \Lambda$ is equivalent to

$$\lim_{x \uparrow r_F} \frac{\overline{F}(x + u/w(x))}{\overline{F}(x)} = \exp(-u), \quad \forall u,$$

where w is a positive scaling function satisfying

$$\lim_{x \uparrow r_F} xw(x) = \infty, \quad \text{and} \quad \lim_{x \uparrow r_F} w(x)(r_F - x) = \infty \quad \text{if } r_F < \infty.$$

Recall that the scaling function w can be defined asymptotically via the mean excess function by

$$w(x) \sim \frac{1}{E(X - x | X > x)}, \quad u \uparrow r_F,$$

where X is a random variable distributed by $F \in \mathrm{MDA}(\Lambda)$.

The Weibull Case

The functional form of the Weibull distribution is $\Psi_\gamma(x) = \exp(-|x|^\gamma)$, $x \le 0$. If $F \in \mathrm{MDA}(\Psi_\gamma)$, then r_F is finite and (5) is equivalent to

$$\lim_{u \uparrow r_F} \frac{\overline{F}(r_F - x/u)}{\overline{F}(r_F - 1/u)} = x^\gamma, \quad \forall x > 0.$$

The Weibull case should be ruled out because $F \in \mathcal{S}$ must have an upper endpoint $r_F < \infty$.

Corollary - The Fréchet Case

Corollary 1. If $F \in \mathrm{MDA}(\Phi_\gamma)$ for some $\gamma > 0$, then relation (4) is simplified to

$$\psi(x;n) \sim \overline{F}(x) \sum_{i=1}^{n} E\left(\prod_{j=1}^{i} ((1-\pi_i)(1+r) + \pi_i Q_j)^{-\gamma} \right). \qquad (6)$$

Sarmanov's distribution

We are going to give a joint distribution for $(Q_1, \ldots, Q_n)$ so that those expectations contained in relation (6) become transparent.

As in Kotz-Balakrishnan-Johnson (2000), we consider an n-dimensional Sarmanov's distribution with a joint density

$$f(x_1, ..., x_n) = \prod_{i=1}^{n} f_i(x_i) \left(1 + \sum_{1 \le j < k \le n} \omega_{jk} \phi_j(x_j) \phi_k(x_k)\right), \quad (7)$$

where f_i are the corresponding marginal densities, ϕ_i are bounded functions and ω_{jk} are real numbers such that

$$\begin{cases} \int_{-\infty}^{\infty} \phi_i(t) f_i(t) \, dt = 0, \text{ for all } i = 1, ..., n, \\ 1 + \sum_{1 \le j < k \le n} \omega_{jk} \phi_j(x_j) \phi_k(x_k) \ge 0, \text{ for all } (x_1, ..., x_n) \in \mathbb{R}^n. \end{cases}$$

Corollary 1 - The Fréchet Case (Cont.)

Assume that the random vector $(Q_1, ..., Q_n)$ follows an n-dimensional Sarmanov's distribution with density (7) in which all functions ϕ_i are identical to ϕ. Thus, Q_i are identically distributed with common density f_Q.

Introduce $m_\gamma(c) = E\left((Q+c)^\gamma\right)$, $s_\gamma(c) = E\left(\phi(Q)(c+Q)^{-\gamma}\right)$. If $\pi_i = \pi$ for every i, then we have

$$\begin{aligned} & E\left(\prod_{j=1}^{i}\left((1-\pi)(1+r)+\pi Q_j\right)^{-\gamma}\right) \\ = \quad & \frac{1}{\pi^{i\gamma}}\left(m_{-\gamma}^{i}(c) + m_{-\gamma}^{i-2}(c)\, s_\gamma^2(c) \sum_{1\le j<k\le i} \omega_{jk}\right). \end{aligned}$$

Corollary 2 - The Gumbel Case

Notice from (2) that if $P\left(Q_i^{-1} > u\right)$ is regularly varying at infinity with index $-\gamma_i$ for some $\gamma_i \in [0, \infty)$ then $P\left(Y_i > \overline{y}_i - 1/u\right)$ is regularly varying at infinity with index $-\gamma_i$, where

$$\overline{y}_i = (1-\pi_i)^{-1}(1+r)^{-1} \in (0, \infty).$$

Write $x_k = x \prod_{i=1}^{k} 1/\overline{y}_i$.

Corollary 2. Assume that $F \in \mathcal{S} \cap \mathrm{MDA}(\Lambda, w)$ and that Q_i are independent random variables, each with a distribution in $\mathrm{MDA}(\Phi_{\gamma_i})$ for some $\gamma_i \in [0, \infty)$. Then

$$\psi(x;n) \sim \sum_{k=1}^{n} \overline{F}(x_k) \prod_{i=1}^{k} \left[\frac{\Gamma(\gamma_i + 1)}{(\pi_i \overline{y}_i)^{\gamma_i}} P(Q_i > x_k w(x_k)) \right].$$

An Interesting Result - Proposition 1

Proposition 1. Let X and Y be two independent random variables, where X follows a distribution $F \in \mathrm{MDA}(\Lambda, w)$ with an infinite upper endpoint, Y follows a distribution G satisfying

$$\lim_{x\to\infty} \frac{\overline{G}(1-u/x)}{\overline{G}(1-1/x)} = u^{\alpha}, \qquad \forall u > 0,$$

for some $\alpha \geq 0$. Then

$$P\left(XY > x\right) \sim \Gamma(1+\alpha)\overline{G}\left(1 - \frac{1}{xw(x)}\right)\overline{F}(x).$$

Sketch of the Proof of Proposition 1

Write $\overline{G}(1-x) = x^{\alpha} L(1/x)$, where L is slowly varying at infinity. Clearly,

$$P(XY > x) = \int_x^{\infty} \overline{G}(x/y) F(\mathrm{d}y).$$

Substitute $y = x + z/w(x)$ and define the random variable W_x by

$$P(W_x > u) = \frac{\overline{F}(x + u/w(x))}{\overline{F}(x)}, \qquad u \geq 0.$$

Observe that the definition of $F \in \mathrm{MDA}(\Lambda, w)$ means that the limit distribution of W_x as $x \to \infty$ is the exponential distribution with mean 1.

With this notation, we have

$$P(X > x) = \overline{F}(x) E\overline{G}\left(\frac{1}{1 + W_x/(xw(x))}\right).$$

The survival function in this expectation is asymptotically equivalent to

$$W_x^{\alpha}\overline{G}\left(1 - \frac{1}{xw(x)}\right)$$

almost surely as $x \uparrow \infty$.

Then, following some standard arguments of dominated convergence, we obtain the desired result.

Application to Optimal Investment

We show how our result can be used to determine a value of π that maximizes the expected terminal wealth U_n under the constraint that the ruin probability $\psi(x; n)$ is not more than a small positive number, say 5%.

Note that we assume π invariant with i.

Formally, our goal is to determine a value of $\pi \in [0, 1]$ that maximizes the expected value of

$$U_n = \sum_{i=1}^{n} (A_i - B_i) \prod_{j=i+1}^{n} [1 + (1 - \pi)r + \pi Q_j],$$

subject to

$$\psi(x, T) \leq 5\%.$$

Our numerical studies focus on how π varies in the heavy-tail index γ, the correlations of Q_i, and the interest rate r.

Thank you for your attention!!!