The Probability of Absolute Ruin in the Renewal Model with Constant Interest Force

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June, 2010

1. Introduction.

Let us assume that the costs of claims X_k, $k \geq 1$, form a sequence of i.i.d., and non-negative random variables and the inter-occurrence times θ_k , $k \geq 1$, form another sequence of i.i.d. non-negative random variables.
The locations of the successive claims, $T_n = \sum_{k=1}^{n} \theta_k$, constitute a renewal process

$$N(t) = \#\{n \geq 1 : T_n \in [0, t]\} \qquad t \geq 0.$$

Let $c > 0$ be the constant gross premium rate, $\delta > 0$ be the constant interest force, C_t be the aggregate claims and $x > 0$ be the initial surplus of the insurance company. Then the total surplus, denoted by $W_\delta(t)$, satisfies

$$W_\delta(t) = x\, e^{\delta t} + c \int_0^t e^{\delta(t-y)} dy - \int_0^t e^{\delta(t-y)}\, dC_y\,, \qquad t \geq 0. \tag{1}$$

Motivated by the work in Embrechts and Schmidli (1994), we define the ultimate absolute ruin probability as

$$\psi(x, \infty) = P\left(\inf_{t\geq 0} W_\delta(t) < -\frac{c}{\delta} \,\Big|\, W_\delta(0) = x\right), \qquad x \geq 0. \tag{2}$$

Further we define the finite horizon absolute ruin probability as

$$\psi(x, t) = P\left(\inf_{0\leq s\leq t} W_\delta(s) < -\frac{c}{\delta} \,\Big|\, W_\delta(0) = x\right), \qquad x \geq 0. \tag{3}$$

This paper aims at an asymptotic estimate for $\psi(x)$ as the initial surplus x increases for the case where the claim sizes follow a distribution function from the class $\mathcal{S}(\gamma)$ for $\gamma \geq 0$. In doing so, we mainly apply some standard probabilistic arguments in Grey (1994), Cline and Samorodnitsky (1994), and Tang and Tsitsiashvili (2003, 2004).

Write by $W_\delta(T_n)$ the surplus immediately after paying the nth claim, $n \geq 1$.

$$\begin{aligned} W_\delta(T_n) &= W_\delta(T_{n-1})\, e^{\delta\theta_n} + c\int_{T_{n-1}}^{T_n} e^{\delta(T_n - t)} dt - X_n \\ &= W_\delta(T_{n-1})\, e^{\delta\theta_n} + \frac{c}{\delta}\left(e^{\delta\theta_n} - 1\right) - X_n, \qquad n \geq 1. \end{aligned} \tag{4}$$

Introduce

$$V_n = W_\delta(T_n) + \frac{c}{\delta}, \qquad n \geq 0.$$

It follows immediately from (4) that the sequence V_n, $n \geq 1$, satisfies the recurrence equation

$$V_n = V_{n-1} e^{\delta\theta_n} - X_n, \qquad n \geq 1.$$

which further gives that

$$V_n = \left(x + \frac{c}{\delta}\right)\prod_{k=1}^{n} e^{\delta\theta_k} - \sum_{k=1}^{n} X_k \prod_{i=k+1}^{n} e^{\delta\theta_i}, \qquad n \geq 1.$$

We can rewrite the ultimate ruin probability in (2) as

$$\psi(x,\,\infty) = P\left(\inf_{n\geq 1} W_\delta\left(T_n\right) < -\frac{c}{\delta} \,\middle|\, W_\delta\left(0\right) = x\right).$$

With $Y_n = e^{-\delta\theta_n}$, $n \geq 1$, we further rewrite the absolute ruin probability as

$$\begin{aligned}
\psi(x,\,\infty) &= P\left(\inf_{n\geq 1} V_n < 0 \mid V_0 = x + \frac{c}{\delta}\right) \\
&= P\left(\inf_{n\geq 1}\left(\left(x + \frac{c}{\delta}\right) - \sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i\right) < 0\right) \\
&= P\left(\sum_{k=1}^{\infty} X_k \prod_{i=1}^{k} Y_i > x + \frac{c}{\delta}\right).
\end{aligned}$$

We can rewrite the finite-time ruin probability as

$$\begin{aligned}
\psi(x, t) &= P\left(\inf_{1\le n\le N_t} W_\delta\left(T_n\right) < -\frac{c}{\delta} \,\middle|\, W_\delta\left(0\right) = x\right) \\
&= P\left(\inf_{1\le n\le N_t} V_n < 0 \,\middle|\, V_0 = x + \frac{c}{\delta}\right) \\
&= P\left(\inf_{1\le n\le N_t} \left(\left(x + \frac{c}{\delta}\right) - \sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i\right) < 0\right) \\
&= P\left(\sum_{k=1}^{N_t} X_k \prod_{i=1}^{k} Y_i > x + \frac{c}{\delta}\right).
\end{aligned}$$

Proposition 1. *Consider the renewal model with constant interest rate* $\delta > 0$*. With* $Y_n = e^{-\delta\theta_n}$*,* $n \geq 1$*, the ultimate and finite-time absolute ruin probability satisfy the equations*

$$\psi(x,\,\infty) \;=\; P\left(S_\infty > x + \frac{c}{\delta}\right), \qquad (5)$$

$$\psi(x,\,t) \;=\; P\left(S_t > x + \frac{c}{\delta}\right), \qquad (6)$$

for any $u > 0$*, with*

$$S_\infty = \sum_{k=1}^{\infty} X_k \prod_{i=1}^{k} Y_i\,, \quad S_t = \sum_{k=1}^{N_t} X_k \prod_{i=1}^{k} Y_i\,.$$

We remark that relations (5) are very general since in deriving them no any independence or i.i.d. assumption is used.

Definition 2. *Let us say that a distribution* F *on* $[0, \infty)$ *belongs to the class* $\mathcal{S}(\gamma)$ *for some* $\gamma \geq 0$ *if*

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = e^{\gamma y} \tag{7}$$

for any $y \in \mathbb{R}$ *and the limit*

$$\lim_{x\to\infty} \frac{\overline{F^{2*}(x)}}{\overline{F}(x)} = 2 \int_{0-}^{\infty} e^{\gamma y}\, F(dy)\,, \tag{8}$$

exists and is finite.

A larger class, $\mathcal{L}(\gamma)$, is defined by relation (7) alone.
For the well-known subexponential class $\mathcal{S}(0)$, when $\gamma = 0$, the right-hand side of (8) becomes 2.
For two distributions, $F_1 \in \mathcal{L}(\gamma)$ and $F_2 \in \mathcal{L}(\gamma)$, satisfying

$$0 < \liminf \overline{F_1}(x)/\overline{F_2}(x) \leq \limsup \overline{F_1}(x)/\overline{F_2}(x) < \infty \, ,$$

it is known that $F_1 \in \mathcal{S}(\gamma)$ if and only if $F_2 \in \mathcal{S}(\gamma)$; see, e.g. Klüppelberg (1988).
Closely related is the class $\mathcal{R}_{-\infty}$ of distributions with rapidly-varying tails, characterized by the relation

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 0, \qquad y > 1.$$

Clearly, if $F \in \mathcal{L}(\gamma)$ for some $\gamma > 0$ then $F \in \mathcal{R}_{-\infty}$. A lot of distributions in the class $\mathcal{S}(0)$ such as lognormal and Weibull distributions still belong to the class $\mathcal{R}_{-\infty}$.

Theorem 1. *In the compound Poisson model with constant force of interest* $\delta > 0$*, if* $F \in \mathcal{S}(\gamma)$ *for some* $\gamma \geq 0$ *then it holds for every* $0 < t < \infty$ *that*

$$\psi(x,t) \sim \lambda \exp\left\{\frac{\lambda}{\delta}\int_{\gamma \mathrm{e}^{-\delta t}}^{\gamma} \frac{\mathrm{E}\mathrm{e}^{sX}-1}{s}\mathrm{d}s - \frac{\gamma c}{\delta}\right\}\int_0^t \overline{F}\left(x\mathrm{e}^{\delta s}\right)\mathrm{d}s. \tag{9}$$

Lemma 3. *Let* $\{N_t, t \geq 0\}$ *be a Poisson process with arrival times* T_k, $k = 1, 2, \ldots$ *and let* $\{X_k, k = 1, 2, \ldots\}$ *be a sequence of i.i.d. random variables independent of* $\{N_t, t \geq 0\}$*. Given* $N_t = n$ *for arbitrarily fixed* $t > 0$ *and* $n = 1, 2, \ldots,$ *the sum* $\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta T_k}$ *is equal in distribution to the sum* $\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_k}$*, where the random vector* $(U_1, \ldots, U_n)$ *consists of i.i.d. random variables uniformly distributed on* $(0, 1)$ *and is independent of the vector* $(X_1, \ldots, X_n)$*.*

Lemma 4. *For two independent nonnegative random variables* X *and* Y*, if* X *follows a distribution* $F \in \mathcal{S}(\gamma)$ *and* Y *follows a distribution with an upper endpoint*

$$1 = \sup\{y : \Pr(Y \leq y) < 1\},$$

then the product XY *still follows a distribution in the class* $\mathcal{S}(\gamma)$*.*

Lemma 5. *Let F be a distribution on $[0,\infty)$. If $F \in \mathcal{S}(\gamma)$, then (i) it holds for each fixed $n = 1, 2, \ldots$ that*

$$\overline{F^{n*}}(x) \sim n \left(\int_{0-}^{\infty} \mathrm{e}^{\gamma y} F(\mathrm{d}y) \right)^{n-1} \overline{F}(x); \tag{10}$$

(ii) for every $\varepsilon > 0$ there exists some constant $C_\varepsilon > 0$ such that the inequality

$$\frac{\overline{F^{n*}}(x)}{\overline{F}(x)} \leq C_\varepsilon \left(\int_{0-}^{\infty} \mathrm{e}^{\gamma y} F(\mathrm{d}y) + \varepsilon \right)^{n} \tag{11}$$

holds for all $n = 1, 2, \ldots$ and all x.

Proof of Theorem 1. Starting with (6) and conditioning on N_t, we have

$$\psi(x,t) = \sum_{n=1}^{\infty} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta T_k} > x + \frac{c}{\delta} \,\middle|\, N_t = n \right) \Pr\left(N_t = n\right).$$

By means of Lemma 3, we can have a sequence of i.i.d. random variables, $\{U, U_k, k = 1, 2, \ldots\}$, uniformly distributed on the interval $(0, 1)$ and independent of $\{X, X_k, k = 1, 2, \ldots\}$, such that

$$\psi(x,t) = \sum_{n=1}^{\infty} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_k} > x + \frac{c}{\delta} \right) \Pr\left(N_t = n\right).$$

By Lemma 4, the products $X_k \mathrm{e}^{-\delta t U_k}$, $k = 1, 2, \ldots$, are i.i.d. with common distribution from the class $\mathcal{S}(\gamma)$. Therefore by Lemma 5(i), for each $n = 1, 2, \ldots$

$$\Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_k} > x + \frac{c}{\delta}\right) \sim n \mathrm{e}^{-\gamma c/\delta} \left(\mathrm{E}\mathrm{e}^{\gamma X \mathrm{e}^{-\delta t U}}\right)^{n-1} \Pr\left(X \mathrm{e}^{-\delta t U} > x\right).$$

It is tempting to plug in $t = \infty$ on both sides of (9) to get an asymptotic expression for the infinite-time absolute ruin probability. But, the repeated limits with respect to $x \to \infty$ and $t \to \infty$ of the ratio of both sides of (9) may depend on the order of limits, yielding different results. It turns out, however, that this intuitive plug-in result in the compound Poisson model is valid as a consequence of our next main result for the general renewal risk model.

Theorem 2. *In the renewal risk model with constant force of interest $\delta > 0$, if $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$ then*

$$\mathrm{E}\mathrm{e}^{\gamma S_\infty} < \infty, \quad \textit{and} \quad \psi(x, \infty) \sim \mathrm{E}\mathrm{e}^{\gamma S_\infty} \Pr\left(XY > x + \frac{c}{\delta}\right), \tag{12}$$

where $Y \sim \{Y_k = \mathrm{e}^{-\delta\theta_k}, k = 1, 2, \ldots\}$.

Lemma 6. *Let F, F_1, and F_2 be three distributions on $[0,\infty)$ such that $F \in \mathcal{S}(\gamma)$ and that the limit $l_i = \lim_{x\to\infty} \overline{F_i}(x)/\overline{F}(x)$ exists and is finite for $i = 1, 2$. Then*

$$\lim_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F}(x)} = l_1 \int_{0-}^{\infty} \mathrm{e}^{\gamma y} F_2(\mathrm{d}y) + l_2 \int_{0-}^{\infty} \mathrm{e}^{\gamma y} F_1(\mathrm{d}y).$$

Lemma 7. *Let F_1 and F_2 be two distributions on $[0,\infty)$. If $F_1 \in \mathcal{S}(\gamma)$, $F_2 \in \mathcal{L}(\gamma)$, and $\overline{F_2}(x) = O\left(\overline{F_1}(x)\right)$, then $F_1 * F_2 \in \mathcal{S}(\gamma)$ and*

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) \int_{0-}^{\infty} \mathrm{e}^{\gamma y} F_2(\mathrm{d}y) + \overline{F_2}(x) \int_{0-}^{\infty} \mathrm{e}^{\gamma y} F_1(\mathrm{d}y).$$

Proof of Theorem 2. Let Z be a random variable with distribution F and independent of $\{(X, Y), (X_k, Y_k), k = 1, 2, \ldots\}$, and denote the distribution of $Y = \mathrm{e}^{-\delta\theta}$ by G, which is supported on $(0, 1)$. Then,

$$\begin{aligned}
\Pr\left((Z + X)Y > x\right) &= \int_0^1 \Pr\left(Z + X > \frac{x}{y}\right) G(\mathrm{d}y) \\
&\sim 2\mathrm{E}\mathrm{e}^{\gamma X} \int_0^1 \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) \\
&= o(1)\overline{F}(x),
\end{aligned}$$

where the second step is due to $F \in \mathcal{S}(\gamma)$ and the last step due to $F \in \mathcal{R}_{-\infty}$. Therefore, there is some $x_0 > 0$ large enough such that, for all $x > x_0$,

$$\Pr\left((Z + X)Y > x\right) \leq \overline{F}(x). \tag{13}$$

Construct a new conditional random variable $X^* = (Z|Z > x_0)$, whose distribution still belongs to the intersection $\mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$. Then, we can see that

$$(X^* + X)Y \overset{d}{\leq} X^*, \tag{14}$$

or, equivalently, for all x,

$$\Pr\left((X^* + X)Y > x\right) \leq \Pr\left(Z > x | Z > x_0\right). \tag{15}$$

Actually, when $x \leq x_0$ relation (15) is trivial, while when $x > x_0$, by (13),

$$\Pr\left((X^* + X)Y > x\right) \leq \frac{\Pr\left(Z > x\right)}{\Pr\left(Z > x_0\right)} = \Pr\left(Z > x | Z > x_0\right).$$

Thus, relation (15) always holds. Relation (14) leads to

$$(X^* + X_1)Y_1 \overset{d}{\leq} X^*, \qquad (X^* + X_2)Y_2 \overset{d}{\leq} X^*.$$

It follows that

$$((X^* + X_2)\, Y_2 + X_1)\, Y_1 \overset{d}{\leq} X^*.$$

Hence, $S_{T_1} = X_1Y_1 \overset{d}{\leq} X^*$ and $S_{T_2} = X_1Y_1 + X_2Y_2Y_1 \overset{d}{\leq} X^*$. Repeating these iterations we obtain $S_{T_n} \overset{d}{\leq} X^*$ for every $n = 1, 2, \ldots$. Letting $n \to \infty$ yields

$$S_\infty \overset{d}{\leq} X^*,$$

which implies, as a by-product, that $\mathrm{E}e^{\gamma S_\infty} < \infty$. Let $\tilde{S}_\infty$ be a copy of S_∞ independent of $\{(X_k, Y_k), k = 1, 2, \ldots\}$.

Then, for every $n = 1, 2, \ldots$,

$$S_\infty \stackrel{d}{=} S_{T_n} + \tilde{S}_\infty \prod_{i=1}^{n} Y_i.$$

which implies, as a by-product, that $\mathrm{E}e^{\gamma S_\infty} < \infty$. Let $\tilde{S}_\infty$ be a copy of S_∞ independent of $\{(X_k, Y_k), k = 1, 2, \ldots\}$. Then, for every $n = 1, 2, \ldots$,

$$S_\infty \stackrel{d}{=} S_{T_n} + \tilde{S}_\infty \prod_{i=1}^{n} Y_i.$$

Therefore,

$$S_\infty \stackrel{d}{\leq} S_{T_n} + X^* \prod_{i=1}^{n} Y_i.$$

From this we obtain

$$\Pr\left(S_\infty > x\right) \le \int_0^1 \Pr\left(X_1 + \sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i > \frac{x}{y}\right) G(\mathrm{d}y). \tag{16}$$

By Lemma 7,

$$\begin{aligned}
\Pr\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i > x\right) &\le \Pr\left(\left(\sum_{k=2}^{n} X_k + X^*\right) Y_2 > x\right) \\
&\sim \left(\mathrm{E}e^{\gamma X^*}(n-1)\left(\mathrm{E}e^{\gamma X}\right)^{n-2} + \frac{\left(\mathrm{E}e^{\gamma X}\right)^{n-1}}{\overline{F}(x_0)}\right) \int_0^1 \Pr\left(X > \frac{x}{y}\right) G(\mathrm{d}y) \\
&= o(1)\overline{F}(x),
\end{aligned}$$

where the last step is due to $F \in \mathcal{R}_{-\infty}$.

Now we apply Lemma 6 to continue the derivation of (16) to find that

$$\begin{aligned}\Pr\left(S_\infty > x\right) &\lesssim \int_0^1 \mathrm{E}e^{\gamma\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i\right)} \Pr\left(X_1 > \frac{x}{y}\right) G(\mathrm{d}y) \\ &= \mathrm{E}e^{\gamma\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i\right)} \Pr\left(XY > x\right).\end{aligned}$$

Clearly, $\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i$ converges to S_∞ in distribution as $n \to \infty$. Therefore, by the dominated convergence theorem, the expectation on the right-hand side above converges to $\mathrm{E}e^{\gamma S_\infty}$ as $n \to \infty$. This establishes the asymptotic upper bound as

$$\Pr\left(S_\infty > x\right) \lesssim \mathrm{E}e^{\gamma S_\infty} \Pr\left(XY > x\right).$$

It is easier to construct the corresponding asymptotic lower bound. Similarly as above,

$$\begin{aligned}
\Pr\left(S_\infty > x\right) &\geq \Pr\left(S_{T_n} > x\right) \\
&= \int_0^1 \Pr\left(X_1 + \sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i > \frac{x}{y}\right) G(\mathrm{d}y) \\
&\sim \mathrm{E}e^{\gamma\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i\right)} \Pr\left(XY > x\right).
\end{aligned}$$

Clearly, $\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i$ converges to S_∞ in distribution as $n \to \infty$. Therefore, the expectation on the right-hand side above converges to $\mathrm{E}e^{\gamma S_\infty}$ as $n \to \infty$ too. We have

$$\Pr\left(S_\infty > x\right) \gtrsim \mathrm{E}e^{\gamma S_\infty} \Pr\left(XY > x\right).$$

The expectation $\mathrm{E}e^{\gamma S_\infty}$ appearing in relation (12) is generally unknown for $\gamma > 0$. However, if we go back to the compound Poisson model then this quantity is explicitly available, as shown in the following last result:

Theorem 3. *In the compound Poisson model with constant force of interest $\delta > 0$, if $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$ then it holds that*

$$\psi(x, \infty) \sim \lambda \exp\left\{ \frac{\lambda}{\delta} \int_0^\gamma \frac{\mathrm{E}e^{sX} - 1}{s} \mathrm{d}s - \frac{\gamma c}{\delta} \right\} \int_0^\infty \overline{F}\left(x e^{\delta s}\right) \mathrm{d}s. \tag{17}$$

Thank you!