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# A matrix representation of a discrete analogue of Thiele's differential equation for multistate insurance contracts

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# Multistate insurance

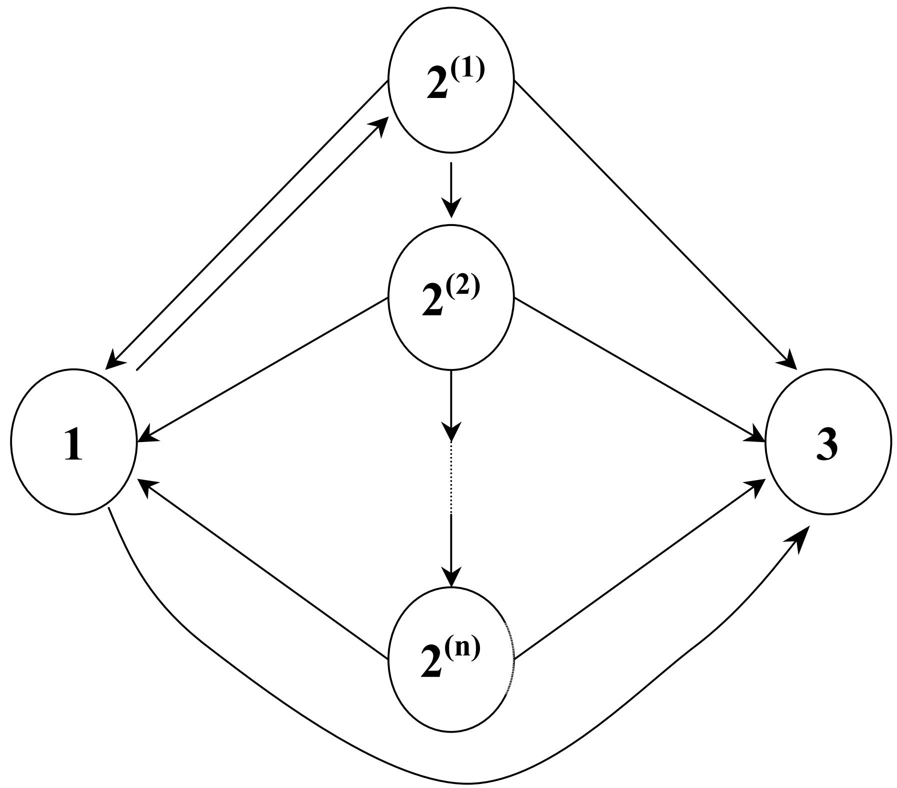
## Example- sickness insurance

$n$ - insurance period

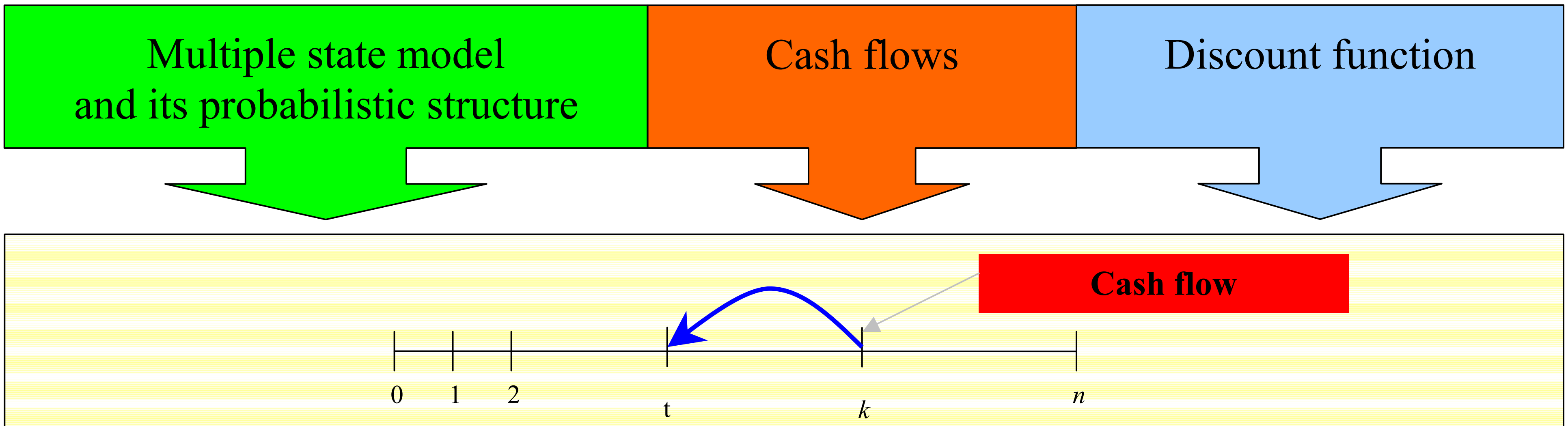
1 - the insured is healthy,

$2^{(h)}$ - the insured is sick with illness duration between $h-1$ and $h$, for $h = 1, 2, \ldots, n$,

3 - the insured is dead.




A disease model with the sickness state split according to duration of illness.




$$ {}_tL = \sum_{k=t}^{n} \sum_{j} \left( \text{discount function}(t,k) \cdot \text{cash flow}(k,j) \right) $$

$$ V_i(t) = \mathrm{E}\left( {}_tL \,\middle|\, \text{at moment } t \text{ contract is at state } i \right) $$

**(Haberman, Pitacco 1999)**

## MULTIPLE STATE MODEL

$$(S,T)$$

**State Space**

$$S=\{1,2,\dots,N\}$$

**$T$:** set of direct transitions between states of the state space,

$(i,j)$ - direct transition from state $i$ to state $j$ ($i \neq j$, $i,j \in S$),

## PROBABILISTIC STRUCTURE OF MODEL

$\{X(t);t\in \mathrm{T}\}$ - stochastic process with values in the finite set $S$.

**ASSUMPTION:** $\mathrm{T}=\{0,1,2,\dots\}$ - discrete, $\{X(t):t=0,1,2,\dots\}$ - *Markov chain*

# INTEREST AND DISCOUNT

- **Stochastic interest rate**:
  - $Y(t)$ - denotes the rate of interest in time interval [0,t];
  - $\{Y(t): t > 0\}$ is a stochastic process with stationary increments.
  - Discount function is of the form $(0 \le t \le k)$

$$\upsilon(t,k) = e^{-(Y(k)-Y(t))}$$

- **Constant interest rate** - discount function is of the form $(0 \le t \le k)$

$$\upsilon(t,k) = \upsilon^{k-t}$$

# Cash Flows

Types of cash flows:

$p_j(k)$ - a period premium amount at time k if $X(k) = j$,

$\pi_j(k)$ - a premium amount at some fixed time $k$ if $X(k) = j$ (for instance $\pi_1(0)$ represents a single premium),

$\ddot{b}_j(k)$ - an annuity due benefit at time $k$ if $X(k) = j$,

$b_j(k)$ - an immediate annuity benefit at time $k$ if $X(k) = j$,

$d_j(k)$ - a lump sum at some fixed time k if $X(k) = j$ (for instance pure endowment),

$c_{ij}(k)$ - a lump sum at time k if a transition occurs from state $i$ to state $j$ at that time (for discrete-time model it means that $X(k) = j$ and $X(k-1) = i$).

Let $\wp$ denote one of the type of cash flows:

$$\wp \in \left\{ p, \pi, \ddot{b}, b, d, c_1, c_2, \ldots, c_N \right\}$$

where $c_i$ is the benefit paid if process $\left\{ X(t) \right\}$ leaves state $i$.

Cash flows

Discount Function

# Present value

Let $\Upsilon_t^{\wp,j}(k)$ be the present value at time *t* of cash flow $\wp$ paied at time *k*, if $X(k)=j$ .

If $\wp \in \{p,\pi,\ddot{b},b,d\}$, then

$$\Upsilon_t^{\wp,j}(k)=\upsilon(t,k)\mathbf{1}_{\{X(k)=j\}}\wp_j(k)$$

If $\wp \in \{c_1,c_2,\ldots,c_N\}$, then

$$\Upsilon_t^{c_i,j}(k)=\begin{cases}\upsilon(t,k)\mathbf{1}_{\{X(k-1)=i\wedge X(k)=j\}}c_{ij}(k) & for \quad i\neq j\\ 0 & for \quad i=j\end{cases}$$

Multiple State Model and its probabilistic structure

Discount Function

Assumptions:

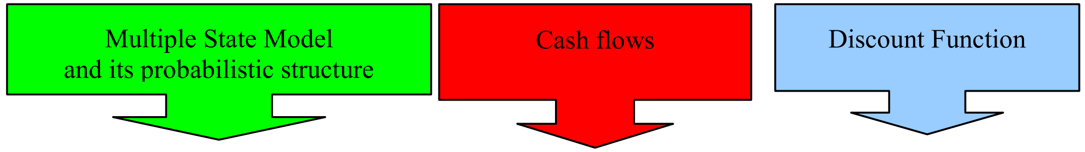
- **Stochastic interest rate**:

**A1** The random variable $X(t)$ is independent of $Y(t)$.

**A2** All moments of random discounting function $e^{-Y(t)}$ are finite.

- **Constant interest rate**:

The discount function is constant.




# Actuarial value

Let $\mathrm{E}\left(\Upsilon_t^{\wp,j}(k) \mid X(t)=i\right)$ be the actuarial value at time $t$ of cash flow $\wp$ paid at time $k$, if $X(k)=j$ and $X(t)=i$ .

$$q_{ij}(t,k) = P\left(X(k)=j \mid X(t)=i\right)$$
$$q_{ij}(t,t+1) = q_{ij}(t)$$

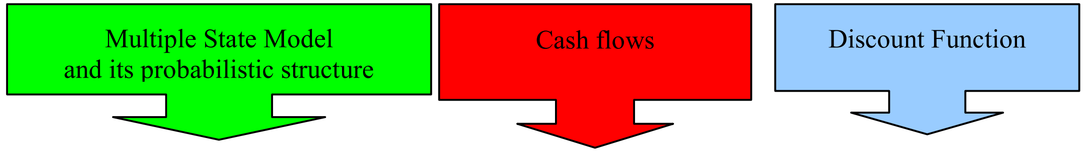
If $\wp \in \left\{p,\pi,\ddot{b},b,d\right\}$, then

$$\mathrm{E}\left(\Upsilon_t^{\wp,j}(k) \mid X(t)=i\right) = \begin{cases} \mathrm{E}\left(\upsilon(t,k)\right) q_{ij}(t,k)\wp_j(k) & for \quad 0 \le t < k \\ \wp_j(k) & for \quad 0 \le t = k \quad \wedge \quad i=j \\ 0 & for \quad 0 \le t = k \quad \wedge \quad i \ne j \end{cases}$$

If $\wp \in \left\{c_1, c_2, \ldots, c_N\right\}$, then

$$\mathrm{E}\left(\Upsilon_t^{c_h,j}(k) \mid X(t)=i\right) =$$

$$\begin{cases} \mathrm{E}\left(\upsilon(t,k)\right) q_{ih}(t,k-1) q_{hj}(k-1,k) c_{hj}(k) & for \quad h \in S \setminus \{j\} \quad \wedge \quad 0 \le t < k \\ q_{hj}(k-1,k) c_{hi}(k) & for \quad h \in S \setminus \{j\} \quad \wedge \quad 0 \le t = k \quad \wedge \quad i=j \\ 0 & besides \end{cases}$$




# Actuarial value

Let $\mathrm{E}\left(\Upsilon_t^{\wp,j}(t_1,t_2) \mid X(t)=i\right)$ be the actuarial value at time $t$ of the sum of cash flows $\wp$ paid in time interval $[t_1,t_2)$, if process $\{X(t)\}$ is at that time in state $j$, condition that $X(t)=i$.

If $\wp \in \{p,\pi,\ddot{b}\}$, then $(0 \le t \le t_1 < t_2)$

$$\mathrm{E}\left(\Upsilon_t^{\wp,j}(t_1,t_2) \mid X(t)=i\right)=\sum_{k=t_1}^{t_2-1}\mathrm{E}\left(\upsilon(t,k)\right)q_{ij}(t,k)\wp_j(k)$$

If $\wp \in \{b,d\}$, then $(0 \le t \le t_1 < t_2)$

$$\mathrm{E}\left(\Upsilon_t^{\wp,j}(t_1,t_2) \mid X(t)=i\right)=\sum_{k=t_1+1}^{t_2}\mathrm{E}\left(\upsilon(t,k)\right)q_{ij}(t,k)\wp_j(k)$$

If $\wp \in \{c_1, c_2, \ldots, c_N\}$, then $(0 \le t \le t_1 < t_2)$

$$\mathrm{E}\left(\Upsilon_t^{c_h,j}(t_1,t_2) \mid X(t)=i\right)=\begin{cases}\sum_{k=t_1+1}^{t_2}\mathrm{E}\left(\upsilon(t,k)\right)q_{ih}(t,k-1)q_{hj}(k-1,k)c_{hj}(k) & for \quad h \in S\setminus\{j\} \\ 0 & for \quad h=j\end{cases}$$

## Prospective loss of the insurer

$$ {}_tL = \sum_{\wp \in \{b,d,c_1,\ldots,c_N\}} \sum_{j \in S} \sum_{k=t+1}^{n} {}_t\Upsilon^{\wp,j}(k) + \sum_{j \in S} {}_t\Upsilon^{\ddot{b},j}(t) - \sum_{\wp \in \{p,\pi\}} \sum_{j \in S} \sum_{k=t}^{n-1} {}_t\Upsilon^{\wp,j}(k) $$

## Prospective reserve

$$ \begin{aligned} V_i(t) &= \mathrm{E}\left( {}_tL \mid X(t) = i \right) \\ &= \sum_{\wp \in \{\ddot{b},b,d,c_1,\ldots,c_N\}} \sum_{j \in S} \mathrm{E}\left( {}_t\Upsilon^{\wp,j}(t,n) \mid X(t) = i \right) - \sum_{\wp \in \{p,\pi\}} \sum_{j \in S} \mathrm{E}\left( {}_t\Upsilon^{\wp,j}(t,n) \mid X(t) = i \right) \end{aligned} $$

**Theorem 1** (*discrete time version of Thiele's differential equation for multistate insurance contract*)

For the insurance contract described by multistate model $(S,T)$, if $Y(t)$ is stochastic process with stationary increments, then

$$V_i(t) = \begin{cases} \mathrm{E}\left(\upsilon\left(t,t+1\right)\right)\sum_{j\in S} q_{ij}(t)\left(V_j(t+1) + \sum_{\wp\in\{b,d,c_i\}} \wp_j(t+1)\right) + \ddot{b}_i(t) - \sum_{\wp\in\{p,\pi\}} \wp_j(t) & \mathit{for} \quad P(X(t)=i)>0 \\ 0 & \mathit{for} \quad P(X(t)=i)=0 \end{cases}.$$

**Remark 1**

If net period premiums are paid when X(t) = 1, then *net premium reserve* is equal

$$V_1(t) = \mathrm{E}\left(\upsilon\left(t,t+1\right)\right)\sum_{j\in S} q_{1j}(t)\left(V_j(t+1) + \sum_{\wp\in\{b,d,c_i\}} \wp_j(t+1)\right) - p_1(t).$$

**Corollary 1 (*premium partition*)**

For the insurance contract described by multistate model $(S,T)$, if $Y(t)$ is stochastic process with stationary increments and net period premiums are paid when X(t) = 1, then net period premium can be presented as follows

$$p_1(t) = p_1^s(t) + \sum_{j\in S\setminus\{1\}} p_1^{r_j}(t)$$

where

$$p_1^s(t) \quad = \quad \mathrm{E}\big(\upsilon(t,t+1)\big)\left(V_1(t+1) + \sum_{\wp\in\{b,d,c_1\}} \wp_1(t+1)\right) + \ddot{b}_1(t) - V_1(t)$$

$$p_1^{r_j}(t) \quad = \quad \mathrm{E}\big(\upsilon(t,t+1)\big)q_{1j}(t)\left(\left(V_j(t+1) + \sum_{\wp\in\{b,d,c_1\}} \wp_j(t+1)\right) - \left(V_1(t+1) + \sum_{\wp\in\{b,d,c_1\}} \wp_1(t+1)\right)\right)$$

**Corollary 1 (*premium partition*)**

For the insurance contract described by multistate model $(S,T)$, if $Y(t)$ is stochastic process with stationary increments and net period premiums are paid when X(t) = 1, then net period premium can be presented as follows

$$\underbrace{p_1(t)}_{} = \underbrace{p_1^{s}(t)}_{\text{saving premium}} + \underbrace{\sum_{j\in S\setminus\{1\}} p_1^{r_j}(t)}_{\text{risk premium}}$$

where

$$p_1^{s}(t) \quad = \quad \mathrm{E}\big(\upsilon(t,t+1)\big)\left(V_1(t+1)+\sum_{\wp\in\{b,d,c_1\}}\wp_1(t+1)\right)+\ddot{b}_i(t)-V_1(t)$$

$$p_1^{r_j}(t) \quad = \quad \mathrm{E}\big(\upsilon(t,t+1)\big)q_{1j}(t)\underbrace{\left(\left(V_j(t+1)+\sum_{\wp\in\{b,d,c_1\}}\wp_j(t+1)\right)-\left(V_1(t+1)+\sum_{\wp\in\{b,d,c_1\}}\wp_1(t+1)\right)\right)}_{nar_j(t)}$$

$nar_j(t)$ ***: net amount at risk for state j at (t+1)th unit time***

$$nar_j(t)=\left(\left(V_j(t+1)+\sum_{\wp\in\{b,d,c_1\}}\wp_j(t+1)\right)-\left(V_1(t+1)+\sum_{\wp\in\{b,d,c_1\}}\wp_1(t+1)\right)\right)\cdot\mathbf{1}_{\{q_{1j}(t)>0\}}$$

# MATRIX REPRESENTATION



$$\mathbf{P}(0) = (1,0,0,\ldots,0)^T \in R^{N^*}$$

$$\mathbf{Q}(\mathrm{t}) = \left[ q_{ij}(t) \right]_{i,j=1}^{N^*} \in R^{N^* \times N^*}$$

where

$$q_{ij}(t) = \mathrm{P}(X^*(t+1) = j \mid X^*(t) = i)$$

$$\mathbf{P}(t) = (pr_1(t), pr_2(t), \ldots, pr_{N^*}(t))^T \in \mathbb{R}^{N^*}$$

where

$$pr_i(t) = \mathrm{P}(X^*(t) = i)$$

$$\mathbf{P}^T(t) = \mathbf{P}^T(0) \prod_{t=0}^{t-1} \mathbf{Q}(t)$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{P}^T(0) \\ \mathbf{P}^T(1) \\ \vdots \\ \mathbf{P}^T(n) \end{pmatrix} \in \mathbb{R}^{N^* \times (n+1)}$$

# MATRIX REPRESENTATION

Discount function

$$\mathbf{\Lambda} = \left[ \lambda_{k_1 k_2} \right]_{k_1, k_2 = 0}^{n} \in R^{(n+1)\times(n+1)}$$

$$\lambda_{k_1 k_2} = \mathrm{E}\left( e^{-\left(Y(k_1) - Y(k_2)\right)} \right) = \begin{cases} \mathrm{E}\left(\upsilon(k_2, k_1)\right) & \text{for} \quad k_1 > k_2 \\ 1 & \text{for} \quad k_1 = k_2 \\ \mathrm{E}\left(r(k_1, k_2)\right) & \text{for} \quad k_1 < k_2 \end{cases}$$

For constant interest rate:

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & \upsilon & \upsilon^2 & \upsilon^3 & \cdots & \upsilon^n \\ \upsilon^{-1} & 1 & \upsilon & \upsilon^2 & \cdots & \upsilon^{n-1} \\ \upsilon^{-2} & \upsilon^{-1} & 1 & \upsilon & \cdots & \upsilon^{n-2} \\ \upsilon^{-3} & \upsilon^{-2} & \upsilon^{-1} & 1 & \cdots & \upsilon^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \upsilon^{-n} & \upsilon^{-(n-1)} & \upsilon^{-(n-2)} & \upsilon^{-(n-3)} & \cdots & 1 \end{pmatrix}$$

# MATRIX REPRESENTATION

Cash flows

$$\begin{pmatrix} cf_1(0) & cf_2(0) & \cdots & cf_N(0) \\ cf_1(1) & cf_2(1) & \cdots & cf_N(1) \\ \vdots & \vdots & \ddots & \vdots \\ cf_1(n) & cf_2(n) & \cdots & cf_N(n) \end{pmatrix} \in R^{N\times(n+1)}$$

**PROBLEM**

$$cf_j(k) = \ddot{b}_j(k) + b_j(k) + d_j(k) + \sum_{i\in S\setminus\{j\}} c_{ij}(k)\mathbf{1}_{\{X(k-1)=i\}} - p_j(k) - \pi_j(k)$$

**(for prospective loss of the insurer)**

**SOLUTION**

Introduction of the *extended multistate model* $\left(S^*, T^*\right)$

1. Verification of multistate model $\left(S, T\right)$
2. Recurrent procedure of creating extended multistate model $\left(S^*, T^*\right)$

1. **Verification of multistate model** $(S,T)$

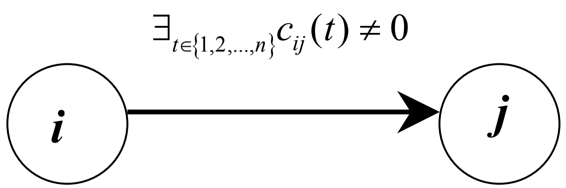
Model $(S,T)$ has to satisfy the following conditions:

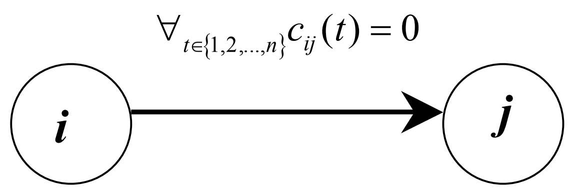
- cash flows $c_{ij}(k)$ do not depend on state *i*

$$c_{ij}(k) = c_j(k),$$

- for each state *j* all transitions to state *j* have to be of the same tape

***cf*** or ***not cf***




*Let* $T^{cf}$ $(T^{cf} \subset T)$ *denote set of pair* $(i,j)$ *of type* ***cf.***

## 2. Recurrent procedure of creating extended multistate model $\left(S^*, T^*\right)$

- If $T^{cf} = \varnothing$, then $\left(S^*, T^*\right) = (S, T)$.
- If $T^{cf} \neq \varnothing$, then in the following $N+1$-steps procedure we construct pair $\left(S^*, T^*\right)$:

***STEP 0***

$$S^* := S$$
$$T^* := T$$
$$T^{cf*} := T^{cf}$$

***STEP j*** $(j = 1, 2, \ldots, N)$

*If there exist* $i \in S^*$ *such that* $(i, j) \in T^{cf*}$ *and* $pr_{jj}(t, t+1) \neq 0$ *(for some* $t \in \{0, 1, \ldots, n-1\}$*, then we associate a new state* $j^+$ *with state j, and*

$$S^* := S^* \cup \left\{j^+\right\},$$

$$T^* := \left(T^* \setminus \left\{(i, j) : i \in S^* \wedge (i, j) \in T^{cf*}\right\}\right) \cup \left\{\left(i, j^+\right) : i \in S^* \wedge (i, j) \in T^{cf*}\right\} \cup \left(j^+, j\right) \cup \left\{\left(j^+, i\right) : i \in S^* \wedge (j, i) \in T^*\right\},$$

$$T^{cf*} := \left\{(i_1, i_2) : i_1, i_2 \in S^* \setminus \{j\} \wedge (i_1, i_2) \in T^{cf*}\right\} \cup \left\{(j, i) : (j, i) \in T^{cf*}\right\} \cup \left\{\left(i, j^+\right) : (i, j) \in T^{cf*}\right\} \cup \left\{\left(j^+, i\right) : (j, i) \in T^{cf*}\right\},$$

*If* $j < N$ *then we go to STEP* $j+1$. *If* $j = N$ *then the procedure is completed.*

# MATRIX REPRESENTATION

Cash flows

$$\begin{pmatrix} cf_1(0) & cf_2(0) & \cdots & cf_N(0) \\ cf_1(1) & cf_2(1) & \cdots & cf_N(1) \\ \vdots & \vdots & \ddots & \vdots \\ cf_1(n) & cf_2(n) & \cdots & cf_N(n) \end{pmatrix} \in R^{N^* \times (n+1)}$$

$$S^* = S \cup S^+$$

$$cf_j(k) = \begin{cases} \ddot{b}_j(k) + b_j(k) + d_j(k) - p_j(k) - \pi_j(k) & for \quad j \in S \\ \ddot{b}_j(k) + b_j(k) + d_j(k) + c_j(k) - p_j(k) - \pi_j(k) & for \quad j \in S^+ \end{cases}$$

$$\mathbf{C}_{in} = \left[ cf_j^{in}(k) \right] \in \mathbb{R}^{N^* \times (n+1)}$$

$$\mathbf{C}_{out} = \left[ cf_j^{out}(k) \right] \in \mathbb{R}^{N^* \times (n+1)}$$

$$\mathbf{C} = \mathbf{C}_{in} + \mathbf{C}_{out}$$

$$\mathbf{C}_{in} = \mathbf{C}_{in^-} + \mathbf{C}_{in_-}$$

For the loss fund of the insurer :

$$cf_j^{in}(k) = \ddot{b}_j(k) + b_j(k) + d_j(k) + c_j(k)$$

$$cf_j^{in^-}(k) = \ddot{b}_j(k)$$

$$cf_j^{in_-}(k) = b_j(k) + d_j(k) + c_j(k)$$

$$cf_j^{out}(k) = -(p_j(k) + \pi_j(k))$$

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$$\mathbf{S} \quad = \quad (1,1,\ldots,1)^T \in \mathbb{R}^{N^*}$$

$$\mathbf{J}_i \quad = \quad \left( 0,0,\ldots,0,\underbrace{1}_{i},0\ldots,0 \right)^T \in \mathbb{R}^{N^*}$$

$$\mathbf{I}_{t+1} \quad = \quad \left( 0,0,\ldots,0,\underbrace{1}_{t+1},0\ldots,0 \right)^T \in \mathbb{R}^{n+1}$$

Let $\mathbf{V}(t)$ be the vector of **prospective reserves** at a given time $t$ for all states of the state space $S$:

$$\mathbf{V}(t) = (V_1(t), V_2(t), \ldots, V_N(t))^T$$

The matrix $\mathbf{V}$ of **prospective reserves for the whole insurance period** is of the form:

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^T(0) \\ \mathbf{V}^T(1) \\ \vdots \\ \mathbf{V}^T(n) \end{pmatrix}$$

**Remark:**

$$V_i^{real}(t) = \begin{cases} V_i(t) & when \quad P(X(t)=i) > 0 \\ - & when \quad P(X(t)=i) = 0 \end{cases}$$

**Theorem 2** (*matrix representation of the vector of prospective reserves*)

For the insurance contract described by extended multistate model $\left(S^*, T^*\right)$ vector of prospective reserves at moment $t$ is in the following form

$$\mathbf{V}(t) = \left(\mathbf{C}_{out}^T + \mathbf{C}_{in^-}^T + \mathbf{F}^T(t,\mathbf{C})\boldsymbol{\Lambda}\right)\mathbf{I}_{t+1}$$

where

$$\mathbf{F}^T(t,\mathbf{C}) = \sum_{k=t+1}^{n}\prod_{u=t}^{k-1}\mathbf{Q}(u)\mathbf{C}^T\mathbf{I}_{k+1}\mathbf{I}_{k+1}^T$$

$\mathbf{F}^T(t,\mathbf{C})$ is the matrix of the **prospective mean future cash flows** counted under condition that at time t process $\{X^*(t)\}$ is at particular state:

$$f_i(t,\mathbf{C}) = \begin{cases} 0 & \text{for} \quad k = 0,1,\ldots,t \\ \sum_{j\in S} \mathrm{P}(X^*(k) = j \mid X^*(t) = i)\cdot cf_j(k) & \text{for} \quad k = t+1, t+2,\ldots,n \end{cases}$$

**Theorem 3** (*matrix representation of Thiele's differential equation for multistate insurance contract*)

For the insurance contract described by extended multistate model $\left(S^*, T^*\right)$, if $Y(t)$ is stochastic process with stationary increments, then

$$\mathbf{V}(t) = \left(\mathbf{Q}(t)\left(\mathbf{V}(t+1) + \mathbf{C}_{in\_}^{T}\mathbf{I}_{t+2}\right)\mathbf{I}_{t+2}^{T}\mathbf{\Lambda}^{T} + \mathbf{C}_{out}^{T} + \mathbf{C}_{in^-}^{T}\right)\mathbf{I}_{t+1}$$

Let $\mathbf{Nar} = \left[ nar_j(t) \right] \in \mathbb{R}^{N^* \times (n+1)}$ be the matrix of **net amount at risk**, where

$$nar_j(t) = \begin{cases} \left( V_j(t+1) + \sum_{\wp \in \{b,d,c_1\}} \wp_j(t+1) \right) - \left( V_1(t+1) + \sum_{\wp \in \{b,d,c_1\}} \wp_1(t+1) \right) & \text{if} \quad q_{1j}(t) > 0 \\ 0 & \text{if} \quad q_{1j}(t) = 0 \end{cases} = \left( \mathbf{J}_j^T - \mathbf{J}_1^T \right) \left( \mathbf{V}^T + \mathbf{C}_{in\_}^T \right) \mathbf{I}_{t+2} \mathbf{1}_{\left\{ \mathbf{J}_1^T Q(t) \mathbf{J}_j > 0 \right\}}$$

**Corollary 2 (*matrix representation of premium partition*)**
For the insurance contract described by extended multistate model $(S^*, T^*)$, if $Y(t)$ is stochastic process with stationary increments and net period premiums are paid when $X^*(t) = 1$, then net period premium can be presented as follows

$$p_1(t) = p_1^s(t) + \sum_{j \in S^* \setminus \{1\}} p_1^{r_j}(t)$$

where

$$p_1^s(t) = \mathbf{J}_1^T \left( \left( \mathbf{V}^T + \mathbf{C}_{in\_}^T \right) \mathbf{I}_{t+2} \mathbf{I}_{t+2}^T \mathbf{\Lambda}^T - \mathbf{V}^T + \mathbf{C}_{in^-}^T \right) \mathbf{I}_{t+1}$$

$$p_1^{r_j}(t) = \mathbf{J}_1^T \mathbf{Q}^*(t) \mathbf{J}_j \mathbf{J}_j^T \mathbf{Nar}^T \mathbf{I}_{t+1} \mathbf{I}_{t+1}^T \mathbf{\Lambda} \mathbf{I}_{t+2}$$

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$$\mathbf{PP} = \begin{pmatrix} p_1^s(0) & p_1^{r_2}(0) & \cdots & p_1^{r_{N^*}}(0) \\ p_1^s(1) & p_1^{r_2}(1) & \cdots & p_1^{r_{N^*}}(1) \\ \vdots & \vdots & \ddots & \vdots \\ p_1^s(n) & p_1^{r_2}(n) & \cdots & p_1^{r_{N^*}}(n) \end{pmatrix}$$

$$p_1^s(t) = \mathbf{I}_{t+1}^T \cdot \mathbf{PP} \cdot \mathbf{J}_1$$

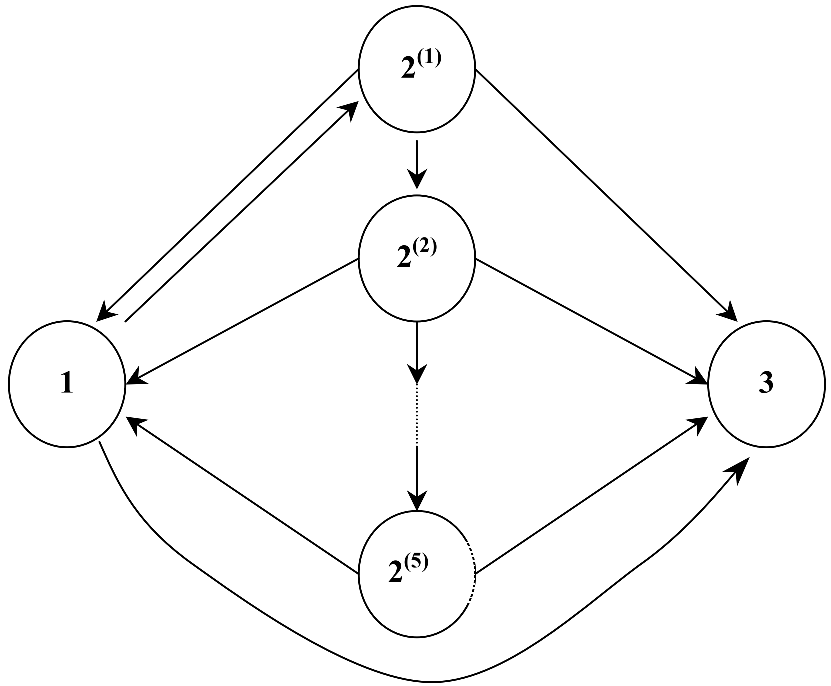
$$p_1^{r_j}(t) = \mathbf{I}_{t+1}^T \cdot \mathbf{PP} \cdot \mathbf{J}_j$$

$$p_1(t) = \mathbf{I}_{t+1}^T \cdot \mathbf{PP} \cdot \mathbf{S}$$

# Example – Sickness insurance

Age at entry: **x=20**
Insurance period : **n=5**
Number of state: **N=7**

A disease model with the sickness state split according to duration of illness.




**State Space:**

1 - the insured is healthy,

$2^{(h)}$ - the insured is sick with illness duration between $h-1$ and $h$, for $h=1,2,\dots,5$,

3 - the insured is dead.

**Benefits:**

$$b_j(k)=\begin{cases} 1 & dla \quad \begin{cases} k=1,2,3,4,5 \wedge j=2^{(1)} \\ k=2,3,4,5 \wedge j=2^{(2)} \\ k=3,4,5 \wedge j=2^{(3)} \\ k=4,5 \wedge j=2^{(4)} \\ k=5 \wedge j=2^{(5)} \end{cases} \\ 0 & poza \quad tym \end{cases}$$

## Matrices related with the distribution of $\{X^*(t)\}$ :

$$\mathbf{Q}(0)=\begin{pmatrix} 0{,}99870657 & 0{,}00023343 & 0 & 0 & 0 & 0 & 0{,}00106000 \\ 0{,}44980000 & 0 & 0{,}50343180 & 0 & 0 & 0 & 0{,}04676820 \\ 0{,}21240000 & 0 & 0 & 0{,}75809905 & 0 & 0 & 0{,}02950095 \\ 0{,}03620000 & 0 & 0 & 0 & 0{,}94479940 & 0 & 0{,}01900060 \\ 0{,}03780000 & 0 & 0 & 0 & 0 & 0{,}94453280 & 0{,}01766720 \\ 0{,}03860000 & 0 & 0 & 0 & 0 & 0{,}94439950 & 0{,}01700050 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Q}(1),\ \mathbf{Q}(2),\ \mathbf{Q}(3),\ \mathbf{Q}(4),\quad \mathbf{P}(0)=(1,0,0,0,0,0,0,0)^{\mathrm{T}}$$

$$\mathbf{P}^T(k)=\mathbf{P}^T(0)\prod_{t=0}^{k-1}\mathbf{Q}(t) \qquad \mathbf{D}=\begin{pmatrix}\mathbf{P}^T(0)\\ \mathbf{P}^T(1)\\ \vdots \\ \mathbf{P}^T(5)\end{pmatrix}$$

# Matrix related with interest rate:

$$\boldsymbol{\Lambda} = \begin{pmatrix} 1 & 1,0400 & 1,0816 & 1,1249 & 1,1699 & 1,2167 \\ 0,9615 & 1 & 1,0400 & 1,0816 & 1,1249 & 1,1699 \\ 0,9246 & 0,9615 & 1 & 1,0400 & 1,0816 & 1,1249 \\ 0,8890 & 0,9246 & 0,9615 & 1 & 1,0400 & 1,0816 \\ 0,8548 & 0,8890 & 0,9246 & 0,9615 & 1 & 1,0400 \\ 0,8219 & 0,8548 & 0,8890 & 0,9246 & 0,9615 & 1 \end{pmatrix}$$

**the annual interest rate:** 4%

# Matrices related with cash flows for loss *fund of the insurer*:

$$\mathbf{C}_{in} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

***Net period premium***

$$\mathbf{C}_{out} = \begin{pmatrix} -p & 0 & 0 & 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$p = \frac{\mathbf{S}^T Diag\left(\mathbf{C}_{in}\mathbf{D}^T\right)\mathbf{\Lambda}\mathbf{I}_1}{\mathbf{I}_1^T\mathbf{\Lambda}^T\left[\mathbf{I} - \sum_{k=m+1}^{n+1}\mathbf{I}_k\mathbf{I}_k^T\right]\mathbf{D}\mathbf{I}_1} = 0{,}000494$$

m = n = 5

$m$ - number of premiums $(1 \le m \le n)$

$$\mathbf{V} = \begin{pmatrix} 0 & 0,26599 & 0,49661 & 0,61867 & 0,61841 & 0,61832 & 0 \\ -0,00007 & 1,48433 & 0,54954 & 0,68130 & 0,68101 & 0,68092 & 0 \\ -0,00013 & 1,20885 & 2,01341 & 0,75017 & 0,74985 & 0,74974 & 0 \\ -0,00017 & 0,88854 & 1,41243 & 1,73472 & 0,82553 & 0,82541 & 0 \\ -0,00015 & 0,51817 & 0,74356 & 0,90897 & 0,90872 & 0,90859 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{V}(t) = \left(\mathbf{C}_{out}^{T} + \mathbf{F}^{T}(t, \mathbf{C})\mathbf{\Lambda}\right)\mathbf{I}_{t+1}$$

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0,99871 & 0,00023 & 0 & 0 & 0 & 0 & 0,00106 \\ 0,99749 & 0,00026 & 0,00012 & 0 & 0 & 0 & 0,00213 \\ 0,99628 & 0,00029 & 0,00014 & 0,00009 & 0 & 0 & 0,00320 \\ 0,99505 & 0,00032 & 0,00015 & 0,00010 & 0,00009 & 0 & 0,00428 \\ 0,99381 & 0,00036 & 0,00017 & 0,00012 & 0,00010 & 0,00008 & 0,00536 \end{pmatrix}$$

$$\mathbf{V}^{real} = \begin{pmatrix} 0 & - & - & - & - & - & - \\ -0,00007 & 1,48433 & - & - & - & - & 0 \\ -0,00013 & 1,20885 & 2,01341 & - & - & - & 0 \\ -0,00017 & 0,88854 & 1,41243 & 1,73472 & - & - & 0 \\ -0,00015 & 0,51817 & 0,74356 & 0,90897 & 0,90872 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_i^{real}(t) = \begin{cases} \mathbf{I}_{t+1}^{T}\mathbf{V}\mathbf{J}_i & when \quad \mathbf{I}_{t+1}^{T}\mathbf{D}\mathbf{J}_i > 0 \\ - & when \quad \mathbf{I}_{t+1}^{T}\mathbf{D}\mathbf{J}_i = 0 \end{cases}$$

## m=5

## net period premium : p= 0.0004940

$$\mathbf{V}^{real} = \begin{pmatrix} 0 & - & - & - & - & - & - \\ -0,00007 & 1,48433 & - & - & - & - & 0 \\ -0,00013 & 1,20885 & 2,01341 & - & - & - & 0 \\ -0,00017 & 0,88854 & 1,41243 & 1,73472 & - & - & 0 \\ -0,00015 & 0,51817 & 0,74356 & 0,90897 & 0,90872 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{Nar} = \begin{pmatrix} 0 & 2,484399 & 0 & 0 & 0 & 0 & 0,000066 \\ 0 & 2,208976 & 0 & 0 & 0 & 0 & 0,000129 \\ 0 & 1,888703 & 0 & 0 & 0 & 0 & 0,000167 \\ 0 & 1,518321 & 0 & 0 & 0 & 0 & 0,000149 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{PS} = \begin{pmatrix} -0,000064 & 0,000558 & 0 & 0 & 0 & 0 & 0,000001 \\ -0,000058 & 0,000552 & 0 & 0 & 0 & 0 & 0,000001 \\ -0,000032 & 0,000525 & 0 & 0 & 0 & 0 & 0,000002 \\ 0,000024 & 0,000470 & 0 & 0 & 0 & 0 & 0,000002 \\ 0,000149 & 0,000345 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

## m=3

## net period premium : p= 0.0007916

$$\mathbf{V}^{real} = \begin{pmatrix} 0 & - & - & - & - & - & - \\ 0,00024 & 1,48470 & - & - & - & - & 0 \\ 0,00050 & 1,20929 & 2,01361 & - & - & - & 0 \\ 0,00080 & 0,88873 & 1,41252 & 1,73474 & - & - & 0 \\ 0,00034 & 0,51817 & 0,74356 & 0,90897 & 0,90872 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{Nar} = \begin{pmatrix} 0 & 2,484455 & 0 & 0 & 0 & 0 & -0,000244 \\ 0 & 2,208790 & 0 & 0 & 0 & 0 & -0,000503 \\ 0 & 1,887934 & 0 & 0 & 0 & 0 & -0,000801 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{PS} = \begin{pmatrix} 0,000234 & 0,000558 & 0 & 0 & 0 & 0 & -0,0000003 \\ 0,000240 & 0,000552 & 0 & 0 & 0 & 0 & -0,0000005 \\ 0,000267 & 0,000525 & 0 & 0 & 0 & 0 & -0,0000008 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$