A multivariate Gaussian ruin problem

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Classical risk process

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k$$

where

u - initial capital

c - premium rate

X_k - i.i.d. claims, $EX_1 = \mu$, $\mathbf{Var}(X_1) = \sigma^2$

$N(t)$ - Poisson process with intensity λ

Diffusion approximation

$$U_n(t) = u + c_n t - \frac{1}{\sqrt{n}} \sum_{k=1}^{N(nt)} X_k$$

If $c_n = c + \lambda\mu\sqrt{n}$, then

$$\{U_n(t)\} \to_{\mathrm{d}} \{u + ct - \sqrt{\lambda}\sigma B(t)\}$$

in $D[0, \infty)$.

- Iglehart

Other models with Gaussian structure

$$U(t) = u + ct - B_H(t)$$

where

$\{B_H(t) : t \geqslant 0\}$ - fractional Brownian motion with hurst parameter $H \in (1/2, 1)$.

- Michna (bad-good periods)

Model

We are interested in the analysis of

$$U(t) = u + d(t) - X(t),$$

where

- $d(t)$ is a deterministic, continuous function (for example $d(t) = ct$)

- $X(t)$ is a centered Gaussian process with continuous sample paths, stationary increments and variance function $\sigma_X^2(t)$ that satisfies

A1 $\sigma_X^2(t) \in C^1[0, \infty)$, convex, strictly increasing

A2 $\sigma_X^2(t)$ is regularly varying at ∞ with parameter $\alpha_\infty \in (0, 2)$ and $\sigma_X^2(t)$ is regularly varying at 0 with parameter $\alpha_0 \in (0, 2]$

Ruin probability

Finite-time ruin probability

$$\psi(u, T) = \mathbb{P}(\inf_{t \in [0,T]} U(t) \leqslant 0)$$

Infinite-time ruin probability

$$\psi(u) = \mathbb{P}(\inf_{t \in [0,\infty)} U(t) \leqslant 0)$$

LDP results for ruin probability

For finite-time problem we have

$$\lim_{u\to\infty} \frac{\log(\psi(u,T))}{u^2} = -\frac{1}{2\sigma_X^2(T)}$$

for general drift $d(t)$.

LDP results for ruin probability

For infinite-time problem we have

$$\lim_{u\to\infty} \frac{\log(\psi(u))}{u^2/\sigma^2(u)} = -2\frac{(2-\alpha_\infty)^{\alpha_\infty-2}}{\alpha_\infty^{\alpha_\infty}}$$

for $d(t) = t$.

- Duffield, O'Connell, KD

Exact asymptotics for finite-time ruin probability

Theorem (Rolski, KD)

Assume that $|d(s) - d(t)| \leqslant \mathrm{Const}|s - t|$. Then

$$\begin{aligned} \psi(u,T) &= \mathbb{P}(X(T) > u + d(T))(1 + o(1)) \\ &= \mathbb{P}\left(\mathcal{N} > \frac{u + d(T)}{\sigma_X(T)}\right)(1 + o(1)) \end{aligned}$$

as $u \longrightarrow \infty$.

Exact asymptotics for infinite-time ruin probability

Assume that $d(t) = ct$. Then

$$\psi(u) = \mathrm{Const}_1 u^{\beta} \mathbb{P}(\mathcal{N} > m(u))$$

with

$$m(u) = \inf_{t \in [0,\infty)} \frac{u + ct}{2\sigma_X(t)}.$$

- Hüsler, Piterbarg, Dieker, KD

Multivariate ruin problem

Let

- $X_1(t), ..., X_n(t)$ - centered Gaussian processes with stationary increments, satisfying $\mathbf{A1\text{-}A2}$;
- Σ_t be the covariance matrix of $(X_1(t), ..., X_n(t))'$;
- $\mathbf{q} = (q_1, ..., q_n)'$

We are interested in the asymptotics of

$$\bar{\psi}(u, \mathbf{q}, T) := \mathbb{P}\left(\exists t \in [0, T] : \bigcap_{i=1}^{n} \{q_i u + d_i(t) - X_i(t) \leqslant 0\}\right)$$

and

$$\bar{\psi}(u, \mathbf{q}) := \mathbb{P}\left(\exists t \in [0, \infty) : \bigcap_{i=1}^{n} \{q_i u + t - X_i(t) \leqslant 0\}\right).$$

Multivariate ruin problem: finite-time horizon

Let

$$I_{\mathbf{X},\mathbf{q}}(T) := \inf_{t\in[0,T]} \inf_{\mathbf{v}\geqslant\mathbf{q}} \left\langle \mathbf{v}, \Sigma_t^{-1}\mathbf{v} \right\rangle .$$

Theorem A Assume that $\det(\Sigma_t) > 0$ for each $t \in [0,T]$. Then

$$\lim_{u\to\infty} \frac{\log\left(\bar{\psi}(u,\mathbf{q},T)\right)}{u^2} = -I_{\mathbf{X},\mathbf{q}}(T).$$

Remark: the asymptotics does not depend on $d(t)$.

Multivariate ruin problem: infinite-time horizon

We focus on a special case: $\mathbf{X}(t) = \mathbf{S}\mathbf{Y}(t)$ with

- $\mathbf{S}$ - an invertible matrix

- $\mathbf{Y}(t) = (Y_1(t), ...Y_n(t))'$ with mutually independent coordinates, $Y_i(t)$ satisfy $\mathbf{A1}$-$\mathbf{A2}$ with $\alpha_{1,\infty} \leqslant ... \leqslant \alpha_{n,\infty}$.

- $d_i(t) = t$

Multivariate ruin problem: infinite-time horizon

We focus on a special case, where $\mathbf{X}(t) = \mathbf{SY}(t)$

Let

- $\mathbf{i}(t) := (t, ..., t)'$
- $\kappa \in \{1, \ldots, n\}$ be the smallest number such that $\lim_{t\to\infty} \sigma_\kappa(t)/\sigma_{\kappa+1}(t) = 0$. Also, for $i \in \{1, \ldots, \kappa\}$, let c_i be such that $\sigma_{i-1}^2 \sim c_i \sigma_i^2$ (set $c_1 = 1$ for $i = 1$).
- $\mathbf{C} := \mathrm{diag}(c_1, \ldots, c_\kappa, 0, \ldots, 0)$.

Theorem B

$$\lim_{u\to\infty} \frac{\log(\bar{\psi}(u, \mathbf{q}))}{u^2/\sigma_{Y_1}^2(u)} = -J(\mathbf{C}, \mathbf{S}, \alpha_1),$$

as $u \to \infty$, where

$$J(\mathbf{C}, \mathbf{S}, \alpha_1) := \inf_{t \geqslant 0} \inf_{\mathbf{v} \geqslant \mathbf{q}} \frac{\langle \mathbf{S}^{-1}(\mathbf{v} + \mathbf{i}(t)), \mathbf{CS}^{-1}(\mathbf{v} + \mathbf{i}(t)) \rangle}{t^{\alpha_1}}.$$

Idea of the proof

◇ Lower bound:

$$\bar{\psi}(u, \mathbf{q}) \geqslant \max_{t \in [0,\infty)} \mathbb{P}\left(\bigcap_{i=1}^{n} \{q_i u + t - X_i(t) \leqslant 0\}\right)$$

Then find uniform bound for the tail of multivariate normal r.v.

Idea of the proof

◇ Upper bound:

For some suitable chosen $w_1, ..., w_n$

$$\bar{\psi}(u, \mathbf{q}) \leqslant \mathbb{P}\left(\exists t \in T : \sum_{i=1}^{n} w_i X_i(t) > \sum_{i=1}^{n} w_i(uq_i + d_i(t))\right)$$

Then we apply Borell inequality. □