

# Optimal Portfolios under Worst Case Scenarios

Carole Bernard<sup>†</sup>, Jit Seng Chen<sup>†</sup> and Steven Vanduffel<sup>§\*</sup>

<sup>†</sup>*Department of Statistics and Actuarial Science, University of Waterloo, Canada.*

<sup>§</sup>*Faculty of Economic, Political and Social Sciences  
and Solvay Business School, Vrije Universiteit Brussel, Belgium.*

*Email:* c3bernar@uwaterloo.ca, jitseng.chen@gmail.com, steven.vanduffel@vub.ac.be

March 2, 2012

**Key-words:** State-dependent preferences, Risk Diversification, Cost-efficiency, Path-dependent Strategies, Growth Optimal Portfolio.

## Summary

Standard portfolio theories such as Mean-Variance optimization, Expected Utility Theory and Cumulative Prospect Theory all assume that investors have law-invariant preferences, i.e. they only look at the distributional properties of terminal wealth and ignore the states of the economy in which the cash-flows are received. Inspired by seminal works of Cox and Leland (1982) and Dybvig (1988), Bernard *et al.* (2011) have shown that law invariant preferences result in optimal strategies that are necessarily decreasing in the stochastic discount factor<sup>1</sup> used for pricing financial instruments. In the context of a Black-Scholes market this is exactly equivalent to saying that optimal payoffs must have a perfect positive correlation with the so-called Growth Optimal Portfolio (GOP). The GOP is a diversified strategy with the property that it almost surely accumulates more wealth than any other strategy at an infinite horizon. This finding elucidates the crucial role of diversification in optimal portfolio selection and provides a solid basis to assert that traditional diversification strategies such as constant-mix strategies and buy-and-hold portfolios are (approximately) optimal for some law-invariant investors. Nevertheless, since the GOP can be generically identified with “*the market*”, this feature of perfect correlation also leads to the conclusion that the worst outcomes for optimal strategies occur in bear markets (e.g. during a financial crisis). Arguably, this does not comply with the aspirations of most investors who may desire some resilience of their wealth against crisis situations.

Hence, we depart from the traditional setting and we investigate the impact of state-dependent preferences on optimal investment decisions. Specifically, we consider investors who target some distribution for their final wealth, but also seek to diversify their investments when there is a crisis. In this respect, we model a crisis as an event where the market index is lower than its Value-at-Risk at some high confidence level. This approach is consistent with the CoVaR approach proposed by Adrian and Brunnermeier (2010) to model a market under stress. We then construct the cheapest (and thus optimal) strategy that delivers a given wealth distribution at the end of the investment horizon and, additionally, preserves a desired dependence with the market in a crisis regime. Comparing the traditional investment strategies such as buy-and-hold and constant-mix portfolio strategies with the newly proposed “optimal” strategies that provide diversification during a market crisis confirms the superiority of the latter under worst-case scenarios.

---

\*S. Vanduffel gratefully acknowledges the financial support of the BNP Paribas Fortis Chair in Banking. C. Bernard and J.S. Chen acknowledge support from TATA Consulting Services. C. Bernard also acknowledges support from the Natural Sciences and Engineering Research Council of Canada.

<sup>1</sup>The stochastic discount factor also appears in the literature as *state price density*, *pricing kernel*, *deflator* or *Radon-Nikodym derivative*.

We believe our results are useful for the following reasons.

*Firstly*, institutional investors typically implement investment strategies that are strongly interconnected with *the market*. This is not a surprise because as stated before, all standard portfolio optimization paradigms (exhibiting law-invariant preferences) necessarily lead to such strategies. Unfortunately, in these settings optimal strategies do not show resilience against bear markets but rather reinforce downward trends. In contrast, our results allow these investors to construct strategies that provide protection under worst-case scenarios for the market. For any given final distribution and any given desired tail dependence with *the market*, we are able to construct the cheapest possible strategy achieving this.

*Secondly*, the strategies we propose do not incur their biggest losses in the worse states in the world, and we can argue they can be used to reduce systemic risk. Our idea of assessing risk and performance of a portfolio not only by looking at its final distribution but also by looking at its interaction with the economic conditions is indeed related to the increasing concern to evaluate systemic risk. Acharya (2009) explains that regulators should “be regulating each bank as a function of both its joint (correlated) risk with other banks as well as its individual (bank-specific) risk”. An insight of this work is that if all institutional investors implement strategies that are resilient against crisis regimes, as we propose, then systemic risk can be diminished. This is because our analysis shows there are many ways to obtain the diversification in the tail for individual investment positions.

*Thirdly*, the ongoing financial crisis with investor and client sentiments reflecting a more risk averse nature further enhances potential interest for strategies that offer protection against bearish markets while keeping some upside potential. In practice, one often resorts to rather complex strategies such as CPPI’s or other structured products to achieve this goal. Here, we provide rather simple alternatives in the sense that they only involve two assets at the maximum. In this sense, our work contributes further to the demystification of complex products and contributes to a better insight in their properties. This was already investigated in Dybvig (1988a,b) and Bernard *et al.* (2011) in the context of law-invariant preferences, but here we extend the setting to better reflect real-world preferences.

*Fourthly*, on a more technical note our work contributes to dependence modelling. It is well studied in actuarial circles how full dependence (comonotonicity) between risks with fixed marginal laws generally represents the worst possible situation for the portfolio loss. In the case where (partial) information on the dependence structure is known, the comonotonic upper bound can be sharpened but it is by no means clear how to do this in general. The results of this paper allow to cope with this problem in certain circumstances; see also Bernard *et al.* (2012) for recent work in this direction. We believe this is a further step to better model portfolio losses in presence of partial information. This in turn is important in the context of Solvency II or any other regulation where portfolio risk needs to be assessed.

## References

- Acharya, V. 2009. “A theory of systemic risk and design of prudential bank regulation,” *Journal of Financial Stability*, **5**, 224–255,
- Adrian, T., Brunnermeier, M., 2010. “CoVaR,” *Working paper available at SSRN*.
- Bernard, C., Boyle, P.P., Vanduffel, S., 2011. “Explicit Representation of Cost-Efficient Strategies,” *Working Paper available at SSRN*.
- Bernard, C., Jiang, X., Vanduffel, S., 2012. “Note on Improved Fréchet bounds and model-free pricing of multi-asset options by Tankov (2011),” *Accepted for publication in the Journal of Applied Probability*.
- Cox, J.C., Leland, H., 1982. “On Dynamic Investment Strategies,” *Proceedings of the seminar on the Analysis of Security Prices*, **26**(2), Center for Research in Security Prices, University of Chicago.
- Dybvig, P., 1988. “Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market,” *Review of Financial Studies*, **1**(1), 67-88.