Quadratic Optimization of Smooth Consumption and Optimal Annuity Design

Kenneth Bruhn[1]. Joint work with **Mogens Steffensen**[2]

Abstract

We propose an optimization criterion that yields extraordinary consumption smoothing compared to the well known results of the life-cycle model. Under this criterion we solve the related consumption optimization problem faced by individuals with preferences for intertemporal stability in consumption. The problem is solved both with and without risky investments. We find that the consumption and investment patterns demanded under the optimization criterion is in general offered as annuity benefits from products in the class of 'formula based smoothed investment-linked annuities'.

Keywords: Stochastic Control, Quadratic Optimization Criteria, Consumption Smoothing, Formula Based Smoothed Investment-Linked Annuities.

We are puzzled: "Why are smooth-benefit and fixed annuities so much more popular than Unit-Link annuities with unsmoothed benefits?"[3]

The widely accepted "Life-Cycle Hypothesis" suggest that peoples' consumption is proportional to their wealth (financial and human wealth) at any time during their life-span. This hypothesis is conform with the results of Merton (1969). He solves the continuous-time optimal consumption and investment problem for a time-additive power utility maximizer with constant relative risk aversion, $1-\gamma$, and impatience factor ρ. The investor can investment in a bond with constant interest rate r and a stock with constant excess-return λ and volatility σ. The formal problem is

$$\sup_{c,\pi} \mathbb{E}\Big(\int_0^T \mathrm{e}^{-\rho s} \left[\tfrac{1}{1-\gamma} c_s^{1-\gamma} ds + \tfrac{1}{1-\gamma} X_s^{1-\gamma} d\varepsilon_T(s) \right] \Big), \tag{1}$$

subject to the wealth dynamics

$$dX_t = (r + \pi_t \lambda) X_t dt + \pi_t \sigma X_t dW_t + (l_t - c_t) dt, \ X_0 = x_0, \tag{2}$$

where W is a standard Brownian motion, l is income rate, π is the proportion of wealth invested in the stock and c is the consumption rate. The optimal solution for consumption is $c_t = (X_t + g(t))/f(t)$, where f and g are deterministic decreasing functions, g being human wealth. The optimal investment is $\pi_t X_t = k(X_t + g(t))$ for $k = \lambda/(\sigma^2\gamma$ which in particular means that the consumption dynamics includes the Brownian motion (short term changes in optimal consumption are stochastic).

The "Life-Cycle Hypothesis" leads to consumption smoothing in the sense that life-time income is consumed over the entire life-span of the individual, thereby leading to savings and dissavings periods. It also incorporates consumption smoothing in another sense. When human wealth is big compared to present savings (in early life), savings fluctuations due to investment market returns have little influence on present consumption. In retirement, though, human wealth is low (not existing unless we account for public pension as part of human wealth), and present consumption varies perfectly with the investment market. Taking this serious, people should dissave using Unit Link annuities rather than with profit or fixed annuities.

The fact that people do not buy Unit-Link annuities is partly puzzled in "The Consumption Smoothing Puzzle", which dates back to Hansen and Singleton (1983). They present evidence that observed consumption is much smoother than predicted by the life cycle model.

[1]Corresponding author. Full address: Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark / SEB Pension Denmark, Postboks 100, DK-0900 Copenhagen C, Denmark. E-mail: bruhn@math.ku.dk / kbx@sebpension.dk

[2]Department of Mathematical Sciences, University of Copenhagen. E-mail: mogens@math.ku.dk

[3]Insurance Information Institute (www.iii.org) reports $7.6 billions of individual immediate fixed annuities sales and $0.1 billions of variable annuities in 2010 in the US. The Association of British Insurers Research Department reports that new sales volumes in UK in 2007 consists of 89% conventional annuities (level/fixed and escalating) and 4% Unit Linked/With profit annuities, the rest being Enhanced/Impaired Life.

Various solutions to the puzzle has been proposed, most of these modifying the market assumption underlying the model in Hansen and Singleton (1983). Examples are stochastic income, mean reverting returns or stochastic volatility of returns. Solutions regarding preference modifications has also been proposed, of particular interest for this work is the introduction of internal additive habit formation, see e.g. Sundaresan (1989). Under these preferences, increased consumption smoothing in comparison with the life-cycle hypothesis is achieved also for low (or no) human wealth. The keyword in this connection being intertemporally dependent preferences, such that utility of present consumption depends on past consumption.

This paper adds to the understanding of the dependence structure in preferences that implies increased consumption smoothing (especially in retirement). The optimization criterion proposed in this paper, though, does not include intertemporally dependent preferences but instead we incorporate intertemporally dependent consumption. This is modeled by allowing for only weak control of consumption, meaning that only the marginal rate of change in consumption being a control variable. This intertemporal dependence structure in consumption obviously leads to more moderate short term variation in consumption compared to Merton's problem.

In the following we will let a be a control variable for the optimization problem, so that the household can choose the rate of change in consumption due to

$$dc_t = adt, c_0 = c_0. \tag{3}$$

We propose the optimization criterion

$$\inf_a \mathbb{E}\left(\int_0^T \left[\tfrac{1}{2}(a_s - \bar{a}c_s)^2 ds + \tfrac{B}{2}(X_s - \xi c_s)^2 d\varepsilon_T(s) \right] \right), \tag{4}$$

where $\bar{a}$, $B \in \mathbb{R}$ and $\xi \geq 0$ are constants and $\varepsilon_T(\cdot) = \mathbf{1}_{\{\cdot \geq T\}}$.

The interpretation of the instantaneous loss-function, $\frac{1}{2}(a - \bar{a}c)^2$, is that the household is only concerned with changes in their marginal consumption, a. Their objective is to minimize the instantaneous quadratic variation away from the specified target $\bar{a}$ ($\bar{a}$ is thus reflecting time-preferences in the consumption).

The terminal loss-function, $\frac{B}{2}(x - \xi c)^2$, is reflecting the bequest motive of the household. The value of ξ sets the target proportion of the final consumption rate that the household wants to have left at time T, and quadratic variation of the savings away from this target is punished. The punishment is weighted by B. A natural example is $\xi = 0$, where the household have no bequest motive and will aim at spending all their wealth.

Being restricted to consumption dynamics in the form of (3) could seem an unrealistic restriction to impose in a model for optimal consumption decisions. There is obviously less control of future consumption under this restriction than seen in classical models. Since this restriction still allows for a consumption pattern consistent with benefits from smoothed annuities, we do not see this as a draw back for the model.

We solve the proposed model (equations (4)+(2)+(3)) both with and without stochastic investments. The derived solutions shows remarkably conformity with the characteristics of annuity products from a particular class of products, namely "Formula Based Smoothed Investment-Linked Annuities". We establish this similarity by comparison of the derived results with a specific Danish product from the class.

References

Hansen, L. P. and Singleton, K. J. (1983). Stochastic consumption, risk aversion, and the temporal behavior of asset returns, *Journal of Political Economy* **91**(2), pp. 249–265.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case, *The Review of Economics and Statistics* **51**(3), 247–57.

Sundaresan, S. M. (1989). Intertemporally dependent preferences and the volatility of consumption and wealth, *Review of Financial Studies* **2**(1), 73–89.