Valuation of Life Insurance Reserves with Dependent Affine Rates

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The problem of valuation of the life insurance liabilities within the setup of a classic Markov model is a classic and well explored problem in the case of a deterministic interest rate and deterministic transition rates. During the last 15 years, there has been an increasing interest in studying the problem where the interest rate and/or the mortality intensity is a stochastic diffusion process. In the case of only a stochastic interest rate, the life insurance reserves can be expressed with the usual formulae if the interest rate is replaced with the *forward interest rate* instead. When the stochastic interest rate is modelled as an affine diffusion process, the forward interest rate is found through the Riccati differential equations. A similar result can be obtained for the survival probability with a stochastic mortality intensity, and if the interest rate and mortality rate are independent processes, the results still hold in the combined model. This work extends the results of stochastic interest and/or transition rates to models with dependent continuous affine interest and transition rates.

As a motivating example, consider a life insurance contract in the classic life-death 2-state Markov model. The payments are deterministic functions, with continuous payments $b(t)$ at time t while alive, and a payment $S(t)$ if death occurs at time t. Let the interest rate r and the mortality transition rate μ be modelled as affine, dependent diffusion processes, such that

$$
\begin{aligned}
r(t) &= c_1(t) + \gamma_1(t)^\top X(t), \\
\mu(t) &= c_2(t) + \gamma_2(t)^\top X(t),
\end{aligned}
$$

where $c_i : [0,T] \to \mathbb{R}_+$, $\gamma_i : [0,T] \to \mathbb{R}_+^d$ for $i = 1,2$ are deterministic functions, and $(X(t))_{t\in[0,T]}$ is a d-dimensional continuous affine process. Here T is some finite time horizon.

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The expected present value $V(t)$ at time t is then given by,

$$V(t) = \mathrm{E}\left[\int_t^T e^{-\int_t^u (r(\tau)+\mu(\tau))\,\mathrm{d}\tau} \left(b(u) + \mu(u)S(u)\right)\,\mathrm{d}u \,\middle|\, \mathcal{F}(t) \right]. \tag{1}$$

Based on the Riccati equations, we derive differential equations that describe the solution to (1). Using these, we can define *generalised forward rates* for the interest rate and the mortality rate, $f_t^r(u)$ and $f_t^\mu(u)$ respectively, such that (1) takes the form,

$$V(t) = \int_t^T e^{-\int_t^u \left(f_t^r(\tau)+f_t^\mu(\tau)\right)\mathrm{d}\tau} \left(b(u) + f_t^\mu(u)S(u)\right)\,\mathrm{d}u.$$

It is noted that the generalised forward interest rate f_t^r does not equal the usual forward rate. The work allows for more complicated examples where we can find similar results for any decrement Markov model. (By decrement we mean Markov models where, when leaving a state, you cannot return). For such models, with multiple decrements (e.g. disability models or surrender models) there does not exist any all-purpose forward rates: The obtained forward rates are in general "state-dependent".

The differential equations describing the solution to the simple example (1) can be found based on the work in [3]. The present talk presents another proof of the same differential equations, which allows for weaker regularity conditions and generalisation to life insurance contracts in any decrement Markov model.

References

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