

Minimization of the function of a quadratic functional and application to the optimal portfolio selection.

Zinoviy Landsman, Udi Makov,
 Department of Statistics, University of Haifa,
 E-mail: landsman@stat.haifa.ac.il, makov@stat.haifa.ac.il

Key words: convex minimization, explicit solution, mean-variance model.

In this paper we generalize the result of [3], where the problem of minimization of the square root functional

$$f_1(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x} + \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}}, \quad \lambda > 0, \quad (1)$$

subject to a number of linear constraints, was considered. Here $\boldsymbol{\mu}$, is $n \times 1$ vector and $\Sigma = (\sigma_{ij})_{i,j=1}^n$ is $n \times n$ positive definite matrix. More precisely, let $B = (b_{ij})_{i,j=1}^{m,n}$ be $m \times n$, $m < n$, rectangular matrix of full rank and \mathbf{c} be some $m \times 1$ vector. The cited paper provides the conditions under which the problem of minimization of function (1), subject to the system of affine constraints,

$$B\mathbf{x} = \mathbf{c}, \quad (2)$$

has a solution. In addition an explicit closed form solution is outlined.

The current paper essentially attempts to extend the results of [3] in the following direction. Instead of goal function (1), we consider a more general function

$$f(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x} + \lambda s(\mathbf{x}^T \Sigma \mathbf{x}), \quad \lambda > 0, \quad (3)$$

where $s(x)$ is defined on $[0, \infty)$ and is positive and differentiable on $(0, \infty)$.

Important examples of functional (3) are

$$f(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x} + \lambda (\mathbf{x}^T \Sigma \mathbf{x})^\beta, \quad 1/2 \leq \beta$$

and

$$f(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x} + \lambda_0 \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} + \sum_{k=1}^l \lambda_k (\mathbf{x}^T \Sigma \mathbf{x})^k, \quad \lambda_k > 0, \quad k = 0, 1, \dots, l,$$

(the case $l = 1$, i.e., $f(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x} + \lambda_0 \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} + \lambda_1 \mathbf{x}^T \Sigma \mathbf{x}$ was considered in [4], where the tail mean-variance principle was introduced).

The problem of minimization of (3), subject to system of linear equality constraints, has many applications, among which are those related to risk management in financial economics. In the special case when $s(x) = x$,

$$f(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x} + \lambda \mathbf{x}^T \Sigma \mathbf{x}, \quad (4)$$

the problem reduces to the classical and well documented mean-variance optimal portfolio selection (see [1]). Here vector \mathbf{x} is interpreted as the weights of portfolio return $R = \mathbf{x}^T \mathbf{X}$, where $\mathbf{X} = (X_1, \dots, X_n)^T$ is a vector of random variables-returns with expectations $E\mathbf{X} = (EX_1, \dots, EX_n)^T = -\boldsymbol{\mu}$ and covariation matrix

$$\text{cov}(\mathbf{X}) = E(\mathbf{X} - E\mathbf{X})(\mathbf{X} - E\mathbf{X})^T = \Sigma.$$

Function (4) is simply

$$f(\mathbf{x}) = E(-R) + \lambda \text{Var}(R), \quad (5)$$

where $\text{Var}(R)$ is the variance of R , which is called the variance premium (see [2], Entry: Premium principles). For $s(x) = \sqrt{x}$ function (3) has a special meaning in Actuarial science: it is the standard deviation premium because it can be rewritten as follows

$$f(\mathbf{x}) = -E(R) + \lambda \sqrt{\text{Var}(R)}.$$

In this note, we obtain conditions under which the solution of minimization of (3) exists, and show how the solution can be explicitly computed.

References

- [1] Boyle, P.P, Cox, S.H,...[et. al]. *Financial Economics: With Applications to Investments, Insurance and Pensions*, Actuarial Foundation, Schaumburg (1998).
- [2] *Encyclopedia of Actuarial Science*, Wiley (2004).
- [3] Landsman, Z. (2008). "Minimization of the root of a quadratic functional under a system of affine equality constraints with application to portfolio management". *Journal of Computational and Applied Mathematics*. 2008, **220**, 1-2, 739-748.
- [4] Landsman, Z. "On the tail mean-variance optimal portfolio selection". *Insurance, Mathematics and Economics*, **46**, (2010), 547-553