Multi-Stock Portfolio Optimization under Prospect Theory - Klaas Schulze

Markowitz (1952) initiates modern portfolio theory by trading off risk and expected return of a portfolio in a one period model. Investment practice was revolutionized by his work and his message that assets should be selected in a way that takes the correlation with the other securities into account. His framework provides the theoretical background for the importance of diversification, as he shows that diversification is a powerful mean of risk reduction. The Markowitz model uses a representative investor which maximizes expected utility of mean-variance type. Since his paper, the predominant optimality criterium underlying portfolio choice is expected utility maximization.

Beyond portfolio theory, expected utility theory is also accepted as the standard normative model of rational choice. It builds on the assumption that agents facing risky alternatives evaluate wealth according to final absolute outcomes, and that they are able to objectively assess probabilities. Furthermore agents are usually assumed to be uniformly risk averse. Expected utility has further been applied as a descriptive model of economic behavior under risk. However, there is substantial empirical and experimental evidence that human behavior deviates from the implications of expected utility, e.g. by paradoxa of Allais or Ellsberg-type or by the observation of subjective probabilities. These violation of expected utility axioms, namely the independence axiom, led to the development of several alternative paradigma of choice with higher descriptive power. One of the most prominent alternatives is Kahnemann and Tversky's (1979, 1992) prospect theory and the subsequent cumulative prospect theory (in short CPT). Roughly speaking, CPT incorporates real human decision patterns and psychology into choice behavior. Benartzi and Thaler (1995) find a CPT setting with an investment horizon of one year to be plausible in explaining the famous equity premium puzzle.

This paper analyzes how such a behavioral CPT-agent optimizes her portfolio. This optimization problem has been recently solved by Bernard and Ghossoub (2010) and He and Zhou (2011) in a one-period setting with one riskless and one risky asset. We solve this static problem in the presence of n risky choices, corresponding to a multi-stock economy. This generalization is significant, since several stocks are more realistic and allow for the essential feature of diversification.

Our first main result is a two-fund separation between the riskless bond and a so-called mean-variance-portfolio, which is a fixed mixture of the n risky stocks. All CPT-agents invest into the same mean-variance-portfolio, independent of the individual risk preferences (which only enter the individual amount invested in this mean-variance-portfolio). Our second main result is deriving this mean-variance-portfolio explicitly. The individual optimal portfolio is thus provided explicitly up to solving a one-dimensional problem. Thus we escape the curse of dimensionality and improve the computational tractability. This semi-closed form solution allows further analytical and numerical analysis of portfolio optimization, as e.g. testing model parameters empirically or considering their sensitivities as in He and Zhou (2011) or establishing a connection to performance measures as in Bernard and Ghossoub (2010). It can further provide insights into modeling issues, into guiding practical portfolio investment, and into explaining financial puzzles.

A brief discussion of the related literature can be found in the full version of this paper.

Deriving the two-fund separation and the mean-variance-portfolio explicitly is appealing and maybe expected to some degree, though it is not trivial. Whereas optimal portfolios have so far been derived by maximizing expected utility, which is a concave function; the problem here is the non-concavity and non-smoothness of the prospect value criterium. The non-concavity arises from the partly non-concave 'S-shaped' utility function and the non-trivial probability distortions. Together they pose a major challenge and standard optimization techniques as convex duality and Lagrange multipliers fail. We tackle this challenge in two approaches: First, under the assumption that the optimal portfolio is finite, we derive its semi-closed form. Second, we drop this assumption. This generality comes at the price that the prospect value can indeed explode for large portfolios. This issue of ill-posedness is a shortcoming of prospect theory and it has sometimes been overlooked in the literature. Here prospect theory sets wrong incentives, as the trade-off between gains and losses is not present. The agent strives for an infinite portfolio. By the infinite portfolio he is exposed to infinite risk. As this excessive risk taking is not desirable, our solution is to restrict the risk of the portfolio by an additional risk constraint in terms of a fairly general risk measure. This can be justified as the internal risk assessment of CPT sets wrong incentives and is too weak in this case. In practice, the external risk measure could be enforced by a regulatory instance. The additional risk constraint restricts optimization to a compact set, and we are able to solve ill-posed problems. It turns out that agents keep the same portfolio composition and adjust only their participation when being forced to reduce risk by a binding risk constraint. We discuss in the body of the paper how this can be interpreted in favor of financial regulation.

Our CPT framework is fairly general in all its three key elements. First, we allow for general reference points or targets. We cover fixed reference points, which include the riskless return. The reference point may depend on the amounts the agents chooses to invest, which includes the expected return. It may also depend on any combination of actual asset returns, so stochastic reference points are allowed. This is important in practice as institutional investors are often evaluated against random benchmarks, e.g. stock indices. Second, we consider general distortion functions, as long as the prospect value is well-defined. Third, we allow for general 'S-shaped' value functions. In this CPT framework we derive the two-fund separation and the mean-variance-portfolio.

Subsequently we analyze three common choices of the value function: First, for the prominent original specification by a piecewise power value function, we derive a condition on the power utility exponents, which ensures the well-posedness of the selection problem. Further we derive the optimal portfolio in closed form. Both results are derived in similar form by He and Zhou (2011) in the one-dimensional case. Second, for piecewise linear value functions we find a threshold on loss aversion, which ensures well-posedness. Here, we also provide the optimal portfolio explicitly, and it turns out to be not investing at all. Third, we consider a piecewise exponential value function. According to recent literature, this specification mimics extreme risk aversion and decision on large-scale lotteries better than the piecewise power specification. Here we derive a threshold on loss aversion ensuring well-posedness. This threshold has appealing sensitivities to the exponential utility exponents.

We assume asset returns to follow a multivariate elliptically symmetric distribution. Elliptical distribution have appealing properties and play a central role in portfolio theory. The elliptical class contains numerous common distributions, e.g. normal, t, exponential power, logistic, normal-variance mixtures, and symmetric generalized hyperbolic distributions. In particular the t distributions, featuring fat tails and leptokurtosis, has been proved to have an excellent fit to asset returns throughout the empirical literature. We explain the elliptical distribution in detail in the body of the paper and comment on its accurate fit.

In order to illustrate our analysis we compute the optimal portfolio for empirical returns of five major US stocks. We build on the estimation of Hu and Kercheval (2010) and observe that the total amount agents with a piecewise power value function invest into the five stocks is generally very high. Further we find that this amount is increasing in power utility exponent on gains and decreasing in power exponent on losses, which is consistent with intuition. Moreover, we find that this amount is much higher for normal returns compared to returns following a t distribution. This indicates that piecewise power agents can be very sensitive to the tail behavior of the returns. Finally, we consider the total amount invested by piecewise exponential CPT-agents and it turns out to be generally lower than for piecewise power agents. This could be explained by the high risk aversion to large-scale lotteries of piecewise exponential agents. Again we find that the amount of total investment has appealing sensitivities to exponential utility exponents.

References

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