Variance optimal stopping problem on geometric Lévy processes

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Keywords: variance criterion, quadratic optimal stopping, Lévy processes

Let X be a Lévy process, let $\mathcal{F}$ be the completed filtration generated by X, and let $\mathcal{T}$ denote the set of stopping times with respect to $\mathcal{F}$. The main problem we consider is to find some stopping time $\tau^* \in \mathcal{T}$ so that

$$\sup_{\tau \in \mathcal{T}} \mathbb{V}[e^{X_\tau}] = \mathbb{V}[e^{X_{\tau^*}}]. \tag{1}$$

This problem we denote *the variance problem*. It is distinguished from classical optimal stopping problems as we maximize variance and not expectation. Hence, we cannot rely on results from e.g. [4] and [5] stating that most classical optimal stopping problems may be solved by hitting times when a solution exists. However, variance is a possible risk meassure, and therefore it is interesting to gain inside to how we solve variance based optimal stopping problems. The variance problem we have studied is a way to approach this field.

In [3], the variance problem is solved for various continuous diffusion processes using a method of embedding the problem into the classical optimal stopping problem below. We denote it *the quadratic problem*:

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[(e^{X_\tau} - c)^2] = \mathbb{E}[(e^{X_{\tau^*}} - c)^2], \quad c > 0. \tag{2}$$

The solution for the variance problem we have found for geometric Lévy processes is based on this embedding method as well, and hence we have first solved the quadratic problem. The quadratic problem has some resemblance to the optimal stopping problem presented in [1] and we have solved it with a method similar to the one used in [1].

As found in [3] it holds that if a solution, τ^*, to (2) fulfills

$$\mathbb{E}[e^{X_{\tau^*}}] = c, \tag{3}$$

then it also solves (1).

The processes studied in [3] all have a combination of $\tau^* \in \mathcal{T}$ and $c > 0$ that fulfills both (2) and (3). But some Lévy processes do not. Let $\bar{X}_\infty = \sup_{t \geq 0} X_t$. The problem in finding a combination of τ^* and c that solves both of the desired properties arises from possible discontinuities in the distribution of $\bar{X}_\infty$. However, it is sometimes possible to use the method even when the distribution of $\bar{X}_\infty$ has discontinuities. We have described a criterion for when this is the case, and a criterion for when we may use the method to find an excess boundary solution.

When there is no combination of τ^* and c fulfilling both (2) and (3), we most easily solve the variance problem by introducing randomized stopping times. The concept is to allow stopping decisions to depend not only on the Lévy process, but also on random choices independent of the Lévy process. Mathematically this is done as in [2]. We define a variable, U, which is uniformly distributed on $[0, 1]$ and independent of X, and we augment the filtration with the information on U. Stopping times with respect to the augmented filtration we denote randomized stopping times. We have shown that it is always possible to solve our variance problem with the embedding method if we maximize over the randomized stopping times.

We return to our original variance problem where randomization is not allowed. When a Lévy process has no excess boundary solution, the solution will depend on the jump structure and the drift.

For compound Poisson processes, the randomized stopping times can be mimicked because these processes stay a positive time in zero before they start moving. This positive time is independent of the rest of the behavior of the process and hence it may be used as the independent random information needed.

The processes with non-compound Poisson jump part will move from zero right away and their filtration will not collect any information before the process has moved. If there is no excess boundary solution then there is no stopping time of $\mathcal{T}$ with as high a variance as the randomized solution. However, the filtration grows sufficiently fast that there is a sequence of stopping times from $\mathcal{T}$ for which the variance converges to the variance of the randomized solution.

In the last case with compound Poisson processes with negative drift we cannot use the approximation method because the filtration does not generate sufficient information. Actually, if there is no excess boundary solution then there is a gap between the value function of the variance problem and the variance problem with randomized stopping times.

References

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