Mean-Variance Optimal Portfolios under State-dependent Preferences

Carole Bernard[†], Steven Vanduffel[§*]

[†]*Department of Statistics and Actuarial Science, University of Waterloo, Canada.*

[§]*Faculty of Economics, Vrije Universiteit Brussel, Belgium.*

Email: `c3bernar@uwaterloo.ca, steven.vanduffel@vub.ac.be`

March 8, 2012

Key-words: State-dependent preferences, Markowitz, Cost-efficiency, Maximum Sharpe Ratio, Pricing Kernel.

Summary

It is common in the academic literature to study optimal investment strategies using an objective function that balances "return" and "risk". This idea was introduced by Markowitz (1952), who together with Roy (1952) first proposed a quantitative approach to determine the optimal trade-off between mean (return) and variance (risk). The mean-variance framework has become very influential due to the fact that it combines simplicity with practical applicability. Markowitz's (1952) paper initiated a tremendous amount of further academic research on optimal portfolio selection problems. Most of the existing approaches have in common that the objective function being optimized is "law-invariant" or "state-independent". In other words the objective is solely driven by the distribution of returns and does not depend on the states of the economy in which they are generated. Such preferences include a wide range of behavioral theories, among them expected utility theory, cumulative prospect theory and rank dependent utility theory.

In practice, agents may face state-dependent constraints when making investment decisions. Purchasing a put option may mean that the agent would like to protect an investment in the underlying asset. In this paper we show how it is possible to derive optimal mean-variance efficient strategies under state-dependent constraints, where each such constraint is interpreted as information on the relationship between the outcomes of the strategy and the states of the financial market. Using the theory of copulas, we develop a constructive method to obtain the cheapest investment strategy which preserves, under state dependent preferences, a given mean and variance.

The results in the present paper require that the market is frictionless, arbitrage-free

*C. Bernard acknowledges support from the Natural Sciences and Engineering Research Council of Canada. S. Vanduffel gratefully acknowledges the financial support of the BNP Paribas Fortis Chair in Banking.

and that the agents agree on the same[1] pricing operator. The paper makes the following contributions:

Traditional optimal mean-variance optimization consists in finding the best allocation of assets assuming a buy-and-hold strategy or a constant mix strategy. The question raises then how mean-variance efficient portfolios can be derived when there is no restriction on trading; in other words, when all trading strategies are allowed and available. Of course, allowing for more trading strategies and thus more degrees of freedom will further enhance optimality. The *first contribution* in this paper is to derive optimal strategies explicitly when there are no restrictions on trading. We show that the optimal portfolio is a short position in the stochastic discount factor used for pricing derivatives and a long position in cash. To derive this result we use recent results on cost-efficient strategies, initiated by Cox and Leland (1982) and Dybvig (1988), and recently extended in Bernard, Boyle and Vanduffel (2011).

In the absence of additional constraints, we are then able to compute the maximum Sharpe ratio when all trading strategies are allowed. This could be useful for example to detect fraud and to check whether the reported performance of a strategy is feasible or not. Recall for example that the Sharpe ratio of Madoff's strategy lied far above the maximum Sharpe ratio for feasible strategies (Bernard and Boyle (2009)). In the second part of the paper we extend our study to the case when there is additional information on the strategy, for example on the way it interacts with the financial market or any other benchmark asset. Our *second contribution* is then to derive tighter bounds on the Sharpe ratio. This could be useful to regulators to improve fraud detection or abnormal performance reporting.

Finally, we also observe that the mean-variance efficient portfolio exhibits its worst outcomes exactly in high states for the stochastic discount factor. This is also very intuitive in the sense that the states of a declining market are the most expensive states to insure, which corresponds to the states where the highest values for the discount factor are observed. To summarize, mean-variance efficient portfolios do not offer protection against bear markets but rather reinforce this. In practice, many investors reward strategies that show resilience against declining markets or, more generally, that exhibit some desired dependence with any other source of background risk (which we generically call a benchmark). Our *third contribution* is to derive mean-variance optimal allocation schemes for investors who exhibit state-dependent preferences in the sense that they care about the first two moments of the strategy's distribution *and* additionally aim at obtaining a desired dependence with a benchmark asset.

References

Bernard, C., Boyle, P.P., "Mr. Madoff's Amazing Returns: An Analysis of the Split-Strike Conversion Strategy". Journal of Derivatives, Vol. 17, No. 1, 2009.

Bernard, C., Boyle, P.P., Vanduffel, S., 2011. "Explicit Representation of Cost-Efficient Strategies," Working Paper available at SSRN.

Cox, J.C., Leland, H., 1982. "On Dynamic Investment Strategies," Proceedings of the seminar on the Analysis of Security Prices," 26(2), Center for Research in Security Prices, Chicago.

Dybvig, P., 1988. "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market," Review of Financial Studies, 1(1), 67-88.

Markowitz, H. 1952. "Portfolio selection," Journal of Finance, 7, 77-91.

Roy, A.D. 1952. "Safety First and the Holding of Assets," Econometrica, 20(30),431-449.

[1] A sufficient condition for this to occur is when the market is complete.