Quadratic Optimization of Smooth Consumption and Optimal Annuity Design

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A Puzzle and a Related Question

"Why are smooth-benefit and fixed annuities so much more popular than Unit-Link annuities with unsmoothed benefits?" (related to "the smooth consumption puzzle", Hansen and Singleton (1983)²).

"In an optimal consumption problem, what kind of preferences should we model in order to have a smooth rather than an unsmooth consumption pattern?"

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Wealth dynamics in retirement:

\[ dX_t = (r + \pi_t \lambda) X_t dt + \pi_t \sigma X_t dW_t - c_t dt, \quad X_0 = x_0, \]

Optimization criterion, given \( x_0 \):

\[ \sup_{c, \pi} \mathbb{E}_{x_0} \left( \int_0^T e^{-\rho s} \left[ \frac{1}{1-\gamma} c_s^{1-\gamma} ds + \frac{1}{1-\gamma} X_s^{1-\gamma} d\varepsilon_T(s) \right] \right), \]

Optimal controls:

\[ c_t^* = \frac{X_t}{\text{func}(t)}, \quad \pi_t^* = \frac{\lambda}{\sigma^2 \gamma}. \]

Consumption varies perfectly with wealth.

Short term volatility:

\[ dc_t = \ldots dt + \ldots dW_t, \]

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A Unit-Link Annuity

Savings dynamics:

\[ dX_t = (r + \pi_t \lambda) X_t dt + \pi_t \sigma X_t dW_t - b_t dt, \quad X_0 = x_0, \]

with benefits

\[ b_t = \frac{X_t}{\theta(t)}, \]

\[ \theta(t) = \int_t^T e^{-\int_t^s r^* d\tau} ds. \]

Matches optimal consumption from time-additive power utility (for specific parameter values).

Short term volatility:

\[ db_t = \ldots dt + \ldots dW_t, \]
Quad. Opt. of Smooth Consumption

\[ dX_t = (r + \pi_t \lambda)X_t dt + \pi_t \sigma X_t dW_t - c_t dt, \quad X_0 = x_0, \]

\[ dc_t = a_t dt, \quad c_0 = c_0. \]

Optimization criterion, given \( x_0 \) and \( c_0 \):

\[ \inf_a \mathbb{E}_{x_0, c_0} \left( \int_0^T \left[ \frac{1}{2} (a_s - \bar{a} c_s)^2 ds + \frac{B}{2} (X_s - \xi c_s)^2 d\varepsilon_T(s) \right] \right). \]

For \( \pi = 0 \), Optimal control:

\[ a_t^* = \bar{a} c_t + \frac{g(t)}{f(t)} (X_t - g(t)c_t), \]

with

\[ f(t) = \int_t^T e^{-2r(s-t)} \left[ g(s)^2 ds + \frac{1}{B} d\varepsilon_T(s) \right], \]

\[ g(t) = \int_t^T e^{-(r-\bar{a})(s-t)} \left[ ds + \xi d\varepsilon_T(s) \right]. \]
Figure: Graphs with initial consumption 10% to high. Parameters are $\pi = 0$, $x_0 = 10$, $r = 4\%$, $\bar{a} = 0\%$, $T = 10$, $B = 50$ and $\xi = 0$. The initial consumption is 1.36 (the preferred initial consumption is $x_0/g(0) = 1.24$).
A Formula Based Smoothed Investment-linked Annuity

Split Savings dynamics \((X)\) in Pension account \((P)\) and Equalization account \((U)\):

\[
dP_t = rP_t dt + \alpha U_t dt - b_t, \quad P_0 = X_0, \\
dU_t = (r + \pi_t \lambda)(P_t + U_t) dt + \pi_t \sigma (P_t + U_t) dW_t - \alpha U_t dt - \theta_t^{-1} U_t dt, \quad U_0 = 0. 
\]

Benefits:

\[
b_t = \frac{P_t}{\theta(t)}. 
\]

Benefit dynamics:

\[
 db_t = d \left(\frac{P_t}{\theta(t)}\right) = (r - r^*)(b_t dt + \frac{\alpha U_t}{\theta(t)} dt).
\]
Summing up...

- Merton’s problem yields unit-link annuity
- Our problem yields smoothed annuity

Expect a working paper before Christmas if things go well!
...and Taking off

Thank you for paying attention!

Questions and comments are always welcome!