Operator methods for the solution of Bellman equations

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7th Samos Conference
in Actuarial Science & Finance
May 31 - June 3, 2012

In memoriam Michel Taksar Sept 9, 1949 - Feb 12, 2012
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Operator methods for the solution of Bellman equations
Pythagoras and Aristarchos from Samos

\[ \|(x, y)\|^2 = x^2 + y^2 \]

Sun is the center.
Active research area with objectives: minimizing ruin probability, or maximizing dividend payment, or else; with control of investment, reinsurance, new business, premia, exposure, several of these.


This talk: Methods for solution:

- smoothness of value function, viscosity solutions?
- (universal) numerical methods?
Simplest model for insurance

Lundberg’s risk model (1905):

\[ R(t) = s + ct - X_1 - \ldots - X_{N(t)}, \]

- \( s \) initial surplus,
- \( c \) constant premium rate,
- \( N(t) \) homogeneous Poisson process for occurrence of claims,
- \( X_1, X_2, \ldots \) iid claim sizes,
- \( N(t), t \geq 0 \), and \( X_1, X_2, \ldots \) independent.
Capital market

Logarithmic Brownian motion for stock, index or similar:

\[ dZ(t) = \mu Z(t) dt + \sigma Z(t) dW(t), \]

independence between \( Z(t) \) and \( R(t) \), \( t \geq 0 \), with \( \mu, \sigma > 0 \).

Riskless asset (no longer existing)

\[ dB(t) = rB(t) dt, \ r \geq 0. \]
Simplifications make life easy

... but without them problems are more interesting!

Simplifying assumptions commonly used:

- unlimited short-selling; (Azcue Muler 2009)
- unlimited leverage; (Pablo Nora 2009)
- no transaction costs; (Thonhauser 2011)
- equal interest rate for borrowing and lending;
- no tax;
- ...

Recent work on bounded short-selling and leverage (Belkina H Luo Taksar 2012)
The title of my talk (nailed down a long time ago) contains the topic **operator methods**. This changed during my work for this conference. I started from a concept where the various operators are discussed, compared and evaluated. Then I felt that a method for the proof of the existence of a solution to the HJB equation should be closely related to an efficient numerical algorithm, and the numerical algorithm should be useful for the mentioned existence proof.

Today I will present a numerical method developed by Alireza Edalati, a student at KIT Karlsruhe, which apparently can be used in the above sense.
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**Operator methods for the solution of Bellman equations**
Minimizing ruin probability by control of investment

Insurers collect premia for payment of claims (rare events). They invest the money in equity, real estate, bonds. Without a proper investment strategy, they face additional investment risk.

What is the optimal investment strategy to minimize ruin probability, with unlimited short-selling and leverage?
The HJB equation is

\[ 0 = \sup_A \{ \lambda E[V(s - X) - V(s)] + (c + A\mu)V'(s) + \frac{1}{2}A^2\sigma^2 V''(s) \}. \]

(1)

The maximizer

\[ A(s) = -\frac{\mu}{\sigma^2} \frac{V'(s)}{V''(s)} \]

defines the optimal strategy in feedback form: invest \( A(s) \) when you are in state \( s \). Insert the maximizer:

\[ \lambda E[V(s - X) - V(s)] + cV'(s) = \frac{1}{2} \frac{\mu^2 V'(s)^2}{\sigma^2 V''(s)}. \]
Exponentially distributed claim sizes, $\lambda + 1/2 = ac$

If $X \sim \text{Exp}(a)$ with density $f(x) = a \exp(-ax), x > 0, a > 0$, we have

$$A(s) = \sqrt{\frac{2c}{a}} \sqrt{1 - \exp(-2as)}.$$

Unlimited leverage:

$$A(s)/s \to \infty, s \to 0.$$
optimal amount invested: $\lambda = 1, c = 2$
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Operator methods for the solution of Bellman equations
Leverage

Leverage $A(s)/s$ is unbounded for $s \to 0$:

$$\lim_{s \to 0} \frac{A(s)}{s} = \lim_{s \to 0} \frac{K \sqrt{s}}{s} = \infty.$$ 

Pablo Azcue and Nora Muler (2009) solved the problem in which the amount invested is restricted to

$$0 \leq A(s) \leq s.$$ 

The HJB for this case: (normed with $r = 0$, $\mu = \sigma = 1$)

$$0 = \sup_{0 \leq A \leq s} \left\{ \lambda E[V(s - X) - V(s)] + (c + A)V'(s) + \frac{1}{2}A^2 V''(s) \right\}.$$ 

attained at $A = 0$ or $A = s$ or at $A = -V'(s)/V''(s)$.

Here, $V(s)$ need not be concave, $V''(s) = 0$ possible, even for exponential claims.
Constrained optimal amount invested

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Operator methods for the solution of Bellman equations
Constrained optimal proportion invested

The case $-\infty < A(s) \leq s$ does not have an optimal strategy!
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An interesting behavior we find in the case of limited leverage/shortselling. Here, the amount invested is restricted to

\[-bs \leq A(s) \leq as, \ a > 1, \ b > 0.\]

The optimal amount invested can have two jumps:

\[
A(s) = \begin{cases} 
    as, & 0 \leq s \leq x_1, \\
    -bs, & x_1 < s \leq x_2, \\
    -\frac{\mu V'(s)}{\sigma^2 V''(s)}, & s > x_2.
\end{cases}
\]

Optimal proportion invested, exp claims, $\lambda = 0.09$, $\mu = 0.02$, $\sigma = 0.1$, $a = 1$, $b = 3$, $c = 0.01$, $r = 0.015$, $\exp(1), a = 1, b = 3, c = 0.02, \lambda = 0.09, \mu = 0.02, r = 0.015$.
effect of jumping strategy, exp(1) claims

exp(1) claims, a=1, b=10, p=a when s<0.5 or s>3

surplus at time t

time t

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Operator methods for the solution of Bellman equations
Many methods and many problems

These different control problems have each their own numerical methods.

Different treatment of small/large values of $s$.

Iterations are slow.

Recursions exhibit inaccuracies close to singularities of the value function.
The Method

Euler type method:
Discretisation $V_\Delta(s)$ for the value function $V(s)$ with step size $\Delta$. Approximate $V'(s)$ and $V''(s)$ by differences:

$$V'_\Delta(s) = \frac{V_\Delta(s) - V_\Delta(s - \Delta)}{\Delta},$$

$$V''_\Delta(s) = \frac{V'_\Delta(s) - V'_\Delta(s - \Delta)}{\Delta}.$$ 

Insert into the HJB equation and solve for $V'_\Delta(s)$:
HJB and its recursion, $\mu = 1, \sigma = 1, r = 0$.

$$0 = \sup_{A \in \mathcal{A}(s)} \{ \lambda E[V(s-X) - V(s)] + (c+A)V'(s) + \frac{1}{2}A^2 V''(s) \}, \quad s > 0. $$

$$V'(s) = \inf_{A \in \mathcal{A}(s)} \frac{\lambda \Delta E[V_\Delta(s-\Delta) - V_\Delta(s-X)] + \frac{1}{2}A^2 V'_\Delta(s-\Delta)}{\Delta(c + A - \lambda \Delta) + \frac{1}{2}A^2}. $$

$$V_\Delta(0) = 1, \quad V'_\Delta(0) = \lambda / c. $$

$$s = k\Delta, \quad k = 1, 2, \ldots $$

Constraint $\mathcal{A}(s)$, e.g. $\mathcal{A}(s) = (-\infty, \infty)$ or $\mathcal{A}(s) = [0, as]$ or $\mathcal{A}(s) = [-bs, as]$. 

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Operator methods for the solution of Bellman equations
Properties of the algorithm

- fast
- stable (concerning value function and its derivatives, NOT strategies)
- universal, can be adapted to optimal investment and reinsurance, also for limited XL reinsurance.
- works also in cases without second derivative
- basic properties of $V_\Delta(s)$ can be easily derived from the recursion
Recursion for control of investment and reinsurance

\[ V'_\Delta(s) = \inf_{A \in A(s), M \in \mathcal{M}} \frac{\lambda \Delta E[V_\Delta(s - \Delta) - V_\Delta(s - g(X, M))] + \frac{1}{2} A^2 V'_\Delta(s - \Delta)}{\Delta(c + A - h(M) - \lambda \Delta) + \frac{1}{2} A^2} \]

\[ h(M) = \rho E[X - g(X, M)], \quad 0 \leq g(s, M) \leq s \]

\[ g(s, M) = \min(s, M) \text{ excess of loss reinsurance} \]
A better recursion with Taylor

\[ V''(s) = \frac{(V'(s) - V'(s - \Delta))}{\Delta}, \]
\[ V(s) = V(s - \Delta) + \Delta V'(s - \Delta) + \frac{1}{2} V''(s) \]

yield the following resursion for \( V'(s) \):

\[ \inf_{A \in \mathcal{A}(s)} \frac{\lambda \Delta E[V(s - \Delta) - V(s - X)] + \frac{1}{2} (A^2 + \Delta^2) V'(s - \Delta)}{\Delta (c + A - \frac{1}{2} \lambda \Delta) + \frac{1}{2} A^2}. \]
Example with linear value function, $A(s) = as$

The better recursion gives exact results in very simple examples (for $0 \leq s \leq s_0$):

$$g(s) = 0, \lambda = 1, a = 1, r = 0, \mu = \sigma = 1.$$  

$$V''(s) = 0, V'(s) = \frac{1}{c}, V(s) = 1 + \frac{1}{c}s$$

can be computed exactly from the recursion for the discretization.
The minimum is taken over a small number of values. The plain vanilla problem needs the minimum of a parabola. The problem with constraints needs the appropriate minimum over the three values (a) minimum of the parabola, (b) the lower bound and (c) the upper bound of $A(s)$.

With reinsurance one needs a complete search over $M'$s (there exist efficient procedures now, Henke (2010)) together with the above procedure for $A$. 
Convergence of $V_\Delta(s)$ to $V(s)$

universal proof (?), monotone convergence (?), behavior at singularities (?)

Proofs by case possible for (a) proportional investment and (b) unconstrained investment: along the following steps:

1. $V_\Delta(s), V'_\Delta(s), V''_\Delta(s)$ uniformly bounded on compact sets without Zero.

2. $V_\Delta(s), V'_\Delta(s), V''_\Delta(s)$ uniformly continuous on compact sets without Zero.

3. $V_\Delta(s), V'_\Delta(s), V''_\Delta(s)$ converge pointwise for some subsequence $\Delta \to 0$.

4. limit solves HJB

5. verification argument.
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Operator methods for the solution of Bellman equations
Viscosity solutions?

In the above control problems, the value function is twice continuously differentiable, and so the classical verification argument works using Ito’s lemma. This is surprising in the case with bounded leverage/shortselling since we have jumps in the optimal strategy. In other cases this smoothness is no longer present, we then have only piecewise a continuous second order derivative.

Here are some examples with control of investment and with simultaneous control of investment and reinsurance. In all of these - except the last one - the second derivative is continuous.
2nd derivative, unconstr

exp(1), shifted exp, Erlang(2,1), Pareto(3), US-Pareto

Available surplus vs. second derivative of V(s)

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Operator methods for the solution of Bellman equations
2nd derivative, no lev/short

exp(1), shifted exp, Erlang(2,1), Pareto(3), US-Pareto

available surplus

second derivative of V(s)
continuous!

exp(1), shifted exp, Erlang(2,1), Pareto(3), US-Pareto

second derivative of V(s)

available surplus
2nd derivative, bounded lev/short

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Operator methods for the solution of Bellman equations
Singular when strategy jumps, a=1, b=3
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In insurance control problems, most often we have a continuous second derivative for the value function. In the rare case without a second derivative we usually have some smoothness left, so we can use a simplified version of viscosity solutions. In the following example we have value functions which is piecewise smooth with only one singular point $s_0 > 0$: The function $V(s)$ is twice continuously differentiable in $[0, s_0]$ and $[s_0, \infty)$ $V(s)$ with finite $V''(s_0-)\text{ and } V''(s_0+)$. 

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Operator methods for the solution of Bellman equations
Viscosity solutions

$V(s)$ is a viscosity solution of $HJB(V, s) = 0$, if for each $s_0 \geq 0$ and test functions $\phi_l(s), \phi_u(s) \in C^2_{[0, \infty)}$ (depending on $s_0$) with the following properties:

\[
\phi_l(s) \leq V(s) \leq \phi_u(s), \quad s \geq 0,
\]

\[
\phi_l(s_0) = \phi_u(s_0) = V(s_0),
\]

we have

\[
HJB(\phi_l, s_0) \leq 0 \text{ and } HJB(\phi_u, s_0) \geq 0.
\]

\[
HJB(f, s) = \sup_{A \in A(s)} \left[ \lambda E[f(s - X) - f(s)] + (c + A)f'(s) + \frac{1}{2}A^2 f''(s) \right].
\]
Verification argument for smooth \( V(s) \)

For a smooth solution \( V(s) \) of the HJB equation with the trivial boundary conditions

\[
V(\infty) = 1, \quad V'(0) = \lambda/c V(0), \quad V(s) = 0 \quad \forall s < 0,
\]

consider

\[
H(t, s) = E[V(X^A(t)|X(0) = s],
\]

where \( X^A(t) \) is the process with investment strategy \( A(s) \). For arbitrary admissible strategy \( A(s) \) \( H(t, s) \) is decreasing in \( t \), so

\[
V(s) \geq H(s, t) \rightarrow \mathbb{P}\{X^A(u) \geq 0 \quad \forall u \geq 0\}, \quad t \rightarrow \infty.
\]
Verification argument for smooth $V(s)$

For a maximizer $A(s)$ of HJB the function $H(s, t)$ is increasing in $t$.

$$V(s) \leq H(s, t) \rightarrow \mathbb{P}\{X^A(u) \geq 0 \ \forall \ u \geq 0\}, \ t \rightarrow \infty.$$ 

Proof for monotonicity of $H(s, t)$ with Ito Lemma and HJB equation. Proof for the limit with exit probabilities for the processes $X^A(t)$.

This proof also shows that an optimal strategy is provided via the maximizer of the HJB equation.
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Operator methods for the solution of Bellman equations
Optimal control of investment with two switching constraints:

\[ A(s) = [0, s], \ s \leq s_0, \ A(s) = (-\infty, \infty), \ s > s_0. \]

We expect that the optimal strategy will jump at \( s_0 \), and we will see that the second derivative will have a jump at \( s_0 \). You can easily guess the optimal strategy, and the verification argument can be given using ideas from viscosity solutions.
optimal proportion and second derivative,
\[ a = 1, \quad b = 0, \quad c = 1, \quad r = 0, \quad \lambda = 1, \quad \mu = 1, \quad \sigma = 1 \]
We construct an upper test function $\phi_u(s)$ by constant continuation of the second derivative. With this function we have $\phi_u(s) \geq V(s)$, $\phi_u(s_0) = V(s_0)$ and $HJB(\phi_u, s_0) = 0$. Using $\phi_u(s)$, we can show that $V(s)$ is not smaller than the largest possible survival probability. The proof is exactly as in the case with smooth $V(s)$. Easy part of the verification argument.
We construct test functions $V_\varepsilon(s)$ satisfying the trivial boundary conditions with

$$V(s) - \varepsilon \leq V_\varepsilon(s) \leq V(s) + \varepsilon, \quad s \geq 0,$$

satisfying $HJB(V_\varepsilon, s_0) \geq 0$. Let $A_\varepsilon(s)$ be a maximizer for $HJB(V_\varepsilon, s)$ and $X^A(t)$ the process with investment strategy $A_\varepsilon(s)$ starting at $s_0$. Then $E[V_\varepsilon(X^A(t))]$ is non-decreasing, and so

$$V(s_0) - \varepsilon \leq V_\varepsilon(s_0) \leq E[V_\varepsilon(X^A(t))].$$

$$E[V_\varepsilon(X^A(t))] \to \mathbb{P}\{X^A(u) \geq 0 \quad \forall u \geq 0\}, \quad s \geq 0,$$

implies the assertion.

This does NOT prove that the maximizer in the HJB yields an optimal strategy!
Construction of smooth $V_\epsilon$

Replace $\mathcal{A}(s) = (-\infty, \infty)$ by an equivalent finite interval $[0, B]$. Approximate the step function

$$f(s) = a \text{ for } s \leq s_0, f(s) = B \text{ for } s > s_0$$

by a continuous function $f_\epsilon(s) \leq f(s)$ and let $V_\epsilon(s)$ be the value function of the investment problem with $\mathcal{A}(s) = [0, f_\epsilon(s)]$. These value functions are smooth and $\epsilon-$ close to $V(s)$ whenever $\epsilon$ is small enough.
$HJB(V_\varepsilon, s) \geq 0$

$\mathcal{A}'(s) \subset \mathcal{A}(s), s \geq 0$;

$HJB'(f, s) = \sup_{A \in \mathcal{A}'(s)} \{ ... \}$

$0 = HJB'(V_\varepsilon, s) \leq HJB(V_\varepsilon, s)$. 

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Operator methods for the solution of Bellman equations
constraints and functions $V_\varepsilon$, 2nd derivatives

constraints

2nd derivatives
functions $V_\varepsilon$ and 1st derivatives

test functions $V_\varepsilon$

their 1st derivatives
References

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