Default Risk of a Time-homogeneous Diffusion Model[1]

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Motivations

- The traditional study of default risk usually defines default as the event that the firm value goes below a too low level a. In particular, $a = 0$ in classical ruin theory.

- In real world, however, the firm is also regarded as defaulted if its value constantly stays below a moderately low level b for a certain time period c, even though the hitting to level b does not immediately lead to illiquidity of the firm.

- Economic justifications for such a consideration are the US bankruptcy codes Chapter 7 (Liquidation) and Chapter 11 (Reorganization).

Default subject to two thresholds and one grace period

- Suppose that the value process is modeled by $X = \{X_t, t \geq 0\}$ with $X_0 = x_0$.
- For a real number x, denote by T_x the first hitting time to level x.
- Let $a < b$ and $c > 0$ be three exogenously determined constants, with a interpreted as the liquidity threshold, b as the reorganization threshold and c as the grace period.
- Let $\tau_b(c)$ be the first time when the process X has constantly stayed below level b for c units of time.
- See the graphs for T_a and $\tau_b(c)$.
- The default probability is defined by

$$q(x_0) = q(x_0; a, b, c) = \mathrm{P}^{x_0}\left\{T_a \wedge \tau_b(c) < \infty\right\}. \tag{1.1}$$

- It is sometimes more convenient to start with the non-default probability $p(x_0) = 1 - q(x_0)$.

Some immediate remarks

Letting $a \downarrow -\infty$ and $b = 0$ in (1.1) yields

$$q(x_0; -\infty, 0, c) = \mathrm{P}^{x_0}\left\{\tau_0(c) < \infty\right\},$$

which is recognized as Parisian ruin probability.

Clearly, $q(x_0; a, b, c)$ is decreasing in c. Letting $c \downarrow 0$ in (1.1) yields

$$q(x_0; a, b, +0) = \mathrm{P}^{x_0}\left\{T_b < \infty\right\},$$

while letting $c \uparrow \infty$ yields

$$q(x_0; a, b, \infty) = \mathrm{P}^{x_0}\left\{T_a < \infty\right\}.$$

Hence, the grace period c serves as a bridge connecting the two traditional default probabilities:

$$\mathrm{P}^{x_0}\left\{T_a < \infty\right\} \leq q(x_0; a, b, c) \leq \mathrm{P}^{x_0}\left\{T_b < \infty\right\}$$

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The firm value process

Suppose that the firm value is modeled by a time-homogeneous diffusion process $X = \{X_t, t \geq 0\}$, with dynamics

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t. \tag{2.1}$$

In (2.1):

- $X_0 = x_0$ is the initial wealth,
- $\{W_t, t \geq 0\}$ is a standard Brownian motion (Wiener process),
- and $\mu(\cdot)$ and $\sigma(\cdot) > 0$ are two measurable functions satisfying usual conditions of the existence and uniqueness theorem for the stochastic differential equation (2.1).

Denote by $\{\mathcal{F}_t, t \geq 0\}$ the natural filtration generated by $\{W_t, t \geq 0\}$.

The two-sided exit problem

Define

$$G(x) = \exp\left\{-\int^x \frac{2\mu(y)}{\sigma^2(y)}\mathrm{d}y\right\}, \qquad S(x) = \int^x G(y)\mathrm{d}y.$$

The function $S(\cdot)$ is referred to as the scale function of X. To avoid triviality, we assume that $S(\infty) < \infty$.

It is well known that, for $u < x < v$,

$$\mathrm{P}^x\left\{T_u < T_v\right\} = \frac{\int_x^v G(y)\mathrm{d}y}{\int_u^v G(y)\mathrm{d}y}, \qquad \mathrm{P}^x\left\{T_u > T_v\right\} = \frac{\int_u^x G(y)\mathrm{d}y}{\int_u^v G(y)\mathrm{d}y}. \quad (2.2)$$

Letting $v = \infty$ in second relation in (2.2) yields

$$\mathrm{P}^x\left\{T_u = \infty\right\} = \frac{\int_u^x G(y)\mathrm{d}y}{\int_u^\infty G(y)\mathrm{d}y}. \quad (2.3)$$

The main result

Introduce an auxiliary quantity

$$A(a, b, c) = \lim_{\varepsilon \downarrow 0} \frac{\mathrm{P}^{b-\varepsilon}\{T_b > T_a \wedge c\}}{\varepsilon}. \tag{2.4}$$

It will be proved later that $A(a, b, c)$ exists, is finite and equals the boundary derivative of the solution of a PDE. Hence, its value can be easily determined numerically.

Theorem 2.1 For $a < b \leq x_0$ and $c > 0$, the default probability satisfies

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) \int_b^\infty G(y)\mathrm{d}y + G(b)} \int_{x_0}^\infty G(y)\mathrm{d}y. \tag{2.5}$$

Proof of Theorem 2.1

For $x \geq b$, by the strong Markov property,

$$\begin{aligned} p(x) &= \mathrm{P}^x\{T_a = \infty, \tau_b(c) = \infty\} \\ &= \mathrm{P}^x\{T_b = \infty\} + \mathrm{P}^x\{T_a = \infty, \tau_b(c) = \infty, T_b < \infty\} \\ &= \mathrm{P}^x\{T_b = \infty\} + \mathrm{E}^x\left[\mathrm{P}^x\{T_a = \infty, \tau_b(c) = \infty, T_b < \infty | \mathcal{F}_{T_b}\}\right] \\ &= \mathrm{P}^x\{T_b = \infty\} + \mathrm{P}^x\{T_b < \infty\} p(b). \end{aligned} \tag{2.6}$$

It follows that

$$\begin{aligned} p'_+(b) &= \lim_{\varepsilon \downarrow 0} \frac{p(b+\varepsilon) - p(b)}{\varepsilon} \\ &= q(b) \lim_{\varepsilon \downarrow 0} \frac{\mathrm{P}^{b+\varepsilon}\{T_b = \infty\}}{\varepsilon} \\ &= q(b) \frac{G(b)}{\int_b^\infty G(y)\mathrm{d}y}, \end{aligned} \tag{2.7}$$

where in the last step we used (2.3).

Proof of Theorem 2.1 (Cont.)

Similarly, for $x \in (a, b)$ we have

$$p(x) = \mathrm{P}^x \{T_a = \infty, \tau_b(c) = \infty\} = \mathrm{P}^x \{T_b \leq T_a \wedge c\} \, p(b).$$

By Corollary 3.1 below, the limit $A(a, b, c)$ in (2.4) exists and is finite.

It follows that

$$\begin{aligned} p'_-(b) &= \lim_{\varepsilon \downarrow 0} \frac{p(b) - p(b-\varepsilon)}{\varepsilon} \\ &= p(b) \lim_{\varepsilon \downarrow 0} \frac{\mathrm{P}^{b-\varepsilon} \{T_b > T_a \wedge c\}}{\varepsilon} \\ &= p(b) A(a, b, c). \end{aligned} \tag{2.8}$$

Proof of Theorem 2.1 (Cont.)

By Theorem 4.1 below, the function $p(\cdot)$ is differentiable at b. Thus, the conjunction of (2.7) and (2.8) gives

$$p(b) = \frac{G(b)}{A(a, b, c) \int_b^\infty G(y)\mathrm{d}y + G(b)}. \tag{2.9}$$

Substituting (2.9) into (2.6) and using (2.3), we obtain

$$p(x) = \frac{\int_b^x G(y)\mathrm{d}y}{\int_b^\infty G(y)\mathrm{d}y} + \frac{\int_x^\infty G(y)\mathrm{d}y}{\int_b^\infty G(y)\mathrm{d}y} \times \frac{G(b)}{A(a, b, c) \int_b^\infty G(y)\mathrm{d}y + G(b)}.$$

Thus, relation (2.5) follows from $q(x) = 1 - p(x)$.

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A PDE

Consider the modified two-sided exit probability function

$$\phi(x, t; a, b) = \mathrm{P}^x \{T_b \le T_a \wedge t\}, \qquad a < x < b, t \ge 0.$$

The following theorem establishes a PDE for this function:

Theorem 3.1 Suppose $h(x, t)$ solves

$$h_t(x, t) = \mu(x)h_x(x, t) + \frac{1}{2}\sigma^2(x)h_{xx}(x, t), \qquad a < x < b, t > 0,$$

with the boundary conditions $h(b, t) = 1$ and $h(a, t) = 0$ for $t \ge 0$ while $h(x, 0) = 0$ for $a < x < b$. Then

$$h(x, t) = \phi(x, t; a, b), \qquad a \le x \le b, t \ge 0.$$

Proof of Theorem 3.1

Applying Itö's formula and noticing that $h_s(X_s, t-s) = -h_t(X_s, t-s)$,

$$
\begin{aligned}
& \mathrm{d}h(X_s, t-s) \\
&= -h_t(X_s, t-s)\mathrm{d}s + h_x(X_s, t-s)\mathrm{d}X_s + \frac{1}{2}h_{xx}(X_s, t-s)\sigma^2(X_s)\mathrm{d}s \\
&= h_x(X_s, t-s)\sigma(X_s)\mathrm{d}W_s.
\end{aligned}
$$

For a stopping time τ, we have

$$h(X_{\tau\wedge t}, t-\tau\wedge t) = h(x,t) + \int_0^{\tau\wedge t} h_x(X_s, t-s)\sigma(X_s)\mathrm{d}W_s.$$

This implies that $h(x,t) = \mathrm{E}^x\left[h(X_{\tau\wedge t}, t-\tau\wedge t)\right]$.

Let $\tau = T_a \wedge T_b$. By the boundary conditions, we have

$$
\begin{aligned}
h(x,t) &= \mathrm{E}^x\left[h(X_{\tau,}, t-\tau)1_{\{\tau\le t\}}\right] + \mathrm{E}^x\left[h(X_{t,}, 0)1_{\{\tau>t\}}\right] \\
&= \mathrm{E}^x\left[h(a, t-T_a)1_{\{T_a=\tau\le t\}}\right] + \mathrm{E}^x\left[h(b, t-T_b)1_{\{T_b=\tau\le t\}}\right] + 0 \\
&= 0 + \mathrm{P}^x\left\{T_b \le T_a \wedge t\right\} + 0.
\end{aligned}
$$

Existence and finiteness of D(a,b,c)

By the well-known regularity theory of PDE (see, e.g. Theorem 4.22 of Lieberman (1996)), we immediately have the following:

Corollary 3.1 It holds for every fixed $t > 0$ that $\phi_x(x, t; a, b)\big|_{x=b}$ is finite and continuous with respect to t. In particular,

$$\begin{aligned} A(a, b, c) &= \lim_{\varepsilon \downarrow 0} \frac{\mathrm{P}^{b-\varepsilon}\left\{T_b > T_a \wedge c\right\}}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{1 - \phi(b - \varepsilon, c; a, b)}{\varepsilon} \\ &= \phi_x(x, c; a, b)\big|_{x=b} \end{aligned}$$

exists and is finite.

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Laplace transforms of the two-sided exit times

Suppose that $g_1(\cdot; r)$ and $g_2(\cdot; r)$ are two independent positive solutions of

$$\frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) = rg(x), \qquad r \geq 0,$$

with $g_1(\cdot; r)$ decreasing and $g_2(\cdot; r)$ increasing. Define

$$f(y, z; r) = g_1(y; r)g_2(z; r) - g_1(z; r)g_2(y; r).$$

The Laplace transforms of the two-sided exit times for the diffusion process X were first solved by Darling and Siegert (1953):

Lemma 4.1 For $u \leq x \leq v$ and $r \geq 0$, we have

$$\mathrm{E}^x\left[\mathrm{e}^{-rT_u}; T_u < T_v\right] = \frac{f(x, v; r)}{f(u, v; r)}, \qquad \mathrm{E}^x\left[\mathrm{e}^{-rT_v}; T_v < T_u\right] = \frac{f(u, x; r)}{f(u, v; r)}.$$

On the minimum of two exit times

Lemma 4.2 For all x,

$$\lim_{\varepsilon \downarrow 0} \frac{\mathrm{E}^x \left[T_{x+\varepsilon} \wedge T_{x-\varepsilon} \right]}{\varepsilon^2} = C(x),$$

where $C(x) > 0$ is a finite constant with an explicit expression.

Proof. By Lemma 4.1,

$$\begin{aligned}
\mathrm{E}^x \left[T_{x-\varepsilon}; T_{x-\varepsilon} < T_{x+\varepsilon} \right] &= -\frac{\partial}{\partial r} \mathrm{E}^x \left[\mathrm{e}^{-rT_{x-\varepsilon}}; T_{x-\varepsilon} < T_{x+\varepsilon} \right] \Big|_{r=0} \\
&= -\frac{\partial}{\partial r} \frac{f(x, x+\varepsilon; r)}{f(x-\varepsilon, x+\varepsilon; r)} \Big|_{r=0} = \ldots.
\end{aligned}$$

A similar relation for $\mathrm{E}^x \left[T_{x+\varepsilon}; T_{x+\varepsilon} < T_{x-\varepsilon} \right]$ holds. Therefore,

$$\begin{aligned}
\mathrm{E}^x \left[T_{x+\varepsilon} \wedge T_{x-\varepsilon} \right] &= \mathrm{E}^x \left[T_{x+\varepsilon}; T_{x+\varepsilon} < T_{x-\varepsilon} \right] + \mathrm{E}^x \left[T_{x-\varepsilon}; T_{x-\varepsilon} < T_{x+\varepsilon} \right] \\
&= \text{Taylor's expansion} = \ldots.
\end{aligned}$$

A lemma

Lemma 4.3 $\mathrm{P}^b\left\{\tau_b(c) < T_{b-\varepsilon}\right\} = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$.

Proof. For X starting with b, denote the consecutive down-crossing and up-crossing times of levels b and $b+\varepsilon$ by

$$0 = \gamma_0^- < \gamma_0^+ < \cdots < \gamma_{i-1}^- < \gamma_{i-1}^+ < \gamma_i^+ < \cdots .$$

Then, by the strong Markov property of X and Lemma 4.2,

$$\begin{aligned}
\mathrm{P}^b\left\{\tau_b(c) < T_{b-\varepsilon}\right\} &= \sum_{i=0}^{\infty} \mathrm{P}^b\left\{\gamma_i^- < \tau_b(c) < \gamma_i^+ \wedge T_{b-\varepsilon}\right\} \\
&= \sum_{i=0}^{\infty} \mathrm{E}^b\left[\mathrm{E}^b\left[1_{\left\{\gamma_i^- < \tau_b(c) < \gamma_i^+ \wedge T_{b-\varepsilon}\right\}} \middle| \mathcal{F}_{\gamma_i^-}\right]\right] \\
&= \mathrm{P}^b\left\{\tau_b(c) < T_{b+\varepsilon} \wedge T_{b-\varepsilon}\right\} \sum_{i=0}^{\infty} \mathrm{P}^b\left\{\gamma_i^- < T_{b-\varepsilon}\right\} \\
&\leq \frac{1}{c}\mathrm{E}^b\left[T_{b+\varepsilon} \wedge T_{b-\varepsilon}\right] \sum_{i=0}^{\infty} \left(\mathrm{P}^b\left\{T_{b+\varepsilon} < T_{b-\varepsilon}\right\}\right)^i \\
&= O\left(\varepsilon^2\right).
\end{aligned}$$

Theorem 4.1 The function $p(\cdot)$ is differentiable at b.

Proof. By (2.7), it suffices to show that

$$\lim_{\varepsilon \downarrow 0} \frac{p(b) - p(b-\varepsilon)}{\varepsilon} = q(b) \frac{G(b)}{\int_b^\infty G(y) \mathrm{d}y}. \tag{4.1}$$

Proof of Theorem 4.1 (Cont.)

For small $\varepsilon > 0$, we split $p(b)$ into two parts as

$$\mathrm{P}^b \{\tau_b(c) = \infty, T_{b-\varepsilon} = \infty\} + \mathrm{P}^b \{T_a = \infty, \tau_b(c) = \infty, T_{b-\varepsilon} < \infty\}.$$

By Lemma 4.3, the first term equals

$$\mathrm{P}^b \{T_{b-\varepsilon} = \infty\} - \mathrm{P}^b \{\tau_b(c) < \infty, T_{b-\varepsilon} = \infty\} = \mathrm{P}^b \{T_{b-\varepsilon} = \infty\} + O(\varepsilon^2).$$

With some efforts, we can prove that the second term satisfies

$$\mathrm{P}^b \{T_{b-\varepsilon} < \infty\} p(b-\varepsilon) + o(\varepsilon)$$

It follows that

$$p(b) = \mathrm{P}^b \{T_{b-\varepsilon} = \infty\} + \mathrm{P}^b \{T_{b-\varepsilon} < \infty\} p(b-\varepsilon) + o(\varepsilon),$$

which easily implies (4.1).

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Implicit finite difference method

Recall formula (2.5) for the default probability $q(x_0)$:

$$q(x_0) = \frac{A(a,b,c)}{A(a,b,c)\int_b^\infty G(y)\mathrm{d}y + G(b)} \int_{x_0}^\infty G(y)\mathrm{d}y. \qquad (2.5)$$

The only implicit part is the quantity $A(a, b, c)$.

Theorem 3.1 and Corollary 3.1 enable us to compute $A(a, b, c)$ numerically via a PDE. We use the second-order implicit finite difference method to solve $A(a, b, c)$ for both a Brownian motion and a geometric Brownian motion.

Brownian motion case

Assume $\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$, with $x_0 \geq b$, $\mu, \sigma > 0$. Relation (2.5) reduces to (with $\rho = 2\mu/\sigma^2$)

$$q(x_0) = \frac{A(a,b,c)}{A(a,b,c)+\rho} \mathrm{e}^{-\rho(x_0-b)}.$$

Set parameters to $\mu = 0.1$, $\sigma = 0.25$, $a = 0.1$, $b = 0.2$ and $c = 1$. Then

mesh	$A(0.1, 0.2, 1)$	$q(x_0)$	elapsed time (s)
0.005	8.5534038	$1.3801420 \times e^{-3.2x_0}$	0.505305
0.001	8.4987776	$1.3777311 \times e^{-3.2x_0}$	4.119148
0.0005	8.4919795	$1.3774294 \times e^{-3.2x_0}$	44.813284
0.00025	8.4885830	$1.3772786 \times e^{-3.2x_0}$	342.789096

In the meantime,

$$\mathrm{P}^{x_0}\{T_a < \infty\} = 1.3771278 \times \mathrm{e}^{-3.2x_0},$$
$$\mathrm{P}^{x_0}\{T_b < \infty\} = 1.8964809 \times \mathrm{e}^{-3.2x_0},$$
$$\mathrm{P}^{x_0}\{\tau_b(c) < \infty\} = 0.6932042 \times \mathrm{e}^{-3.2x_0}.$$

Geometric Brownian motion case

Assume $\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$, with $x_0 \geq b$ and $\rho = 2\mu/\sigma^2 > 1$. Relation (2.5) reduces to

$$q(x_0) = \frac{A(a,b,c)}{A(a,b,c) + \rho - 1} \left(\frac{b}{x_0} \right)^{\rho - 1}.$$

Set parameters to $\mu = 0.1$, $\sigma = 0.25$, $a = 0.1$, $b = 0.2$ and $c = 1$. Then

mesh	$A(0.1, 0.2, 1)$	$q(x_0)$	elapsed time (s)
0.005	11.495846	$0.024334255 \times x_0^{-2.2}$	0.460361
0.001	11.145638	$0.024212051 \times x_0^{-2.2}$	4.881175
0.0005	11.101517	$0.024196199 \times x_0^{-2.2}$	37.679470
0.00025	11.079429	$0.024188223 \times x_0^{-2.2}$	287.741042

In the meantime,

$$\mathrm{P}^{x_0}\{T_a < \infty\} = 0.0063095734 \times x_0^{-2.2},$$

$$\mathrm{P}^{x_0}\{T_b < \infty\} = 0.028991187 \times x_0^{-2.2},$$

$$\mathrm{P}^{x_0}\{\tau_b(c) < \infty\} = 0.014533261 \times x_0^{-2.2}.$$

Potential future works

- Consider a more general firm value process of strong Markov property;
- Incorporate (heavy-tailed) jumps into the modeling;
- Consider the more practical finite-time case.
- $\cdots$

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Thank You Very Much!!!