A General Theory of Backward Stochastic Difference Equations

Samuel N. Cohen and Robert J. Elliott

University of Adelaide and University of Calgary
Outline

Dynamic Nonlinear Expectations

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Conclusions
A key question in Mathematical Finance is: *Given a future random payoff $X$, what are you willing to pay today for $X$?*

One could also ask “How **risky** is $X$?”

Various attempts have been made to answer this question. (Expected utility, CAPM, Convex Risk Measures, etc...)

While giving an axiomatic approach to answering this question, we shall outline the theory of “Backward Stochastic Difference Equations.”
Nonlinear Expectations

For some terminal time $T$, we define an ‘$\mathcal{F}_t$-consistent nonlinear expectation’ $\mathcal{E}$ to be a family of operators

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t); t \leq T$$

with

1. (Monotonicity) If $Q^1 \geq Q^2$ $\mathbb{P}$-a.s.,

$$\mathcal{E}(Q^1|\mathcal{F}_t) \geq \mathcal{E}(Q^2|\mathcal{F}_t)$$

2. (Constants) For all $\mathcal{F}_t$-measurable $Q$,

$$\mathcal{E}(Q|\mathcal{F}_t) = Q$$
Nonlinear Expectations

3. (Recursivity) For \( s \leq t \),

\[
\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s)
\]

4. (Zero-One law) For any \( A \in \mathcal{F}_t \),

\[
\mathcal{E}(1_A Q|\mathcal{F}_t) = 1_A \mathcal{E}(Q|\mathcal{F}_t).
\]
Nonlinear Expectations

Two other properties are desirable

5. (Translation invariance) For any $q \in L^2(F_t)$,

$$\mathcal{E}(Q + q | F_t) = \mathcal{E}(Q | F_t) + q.$$

6. (Concavity) For any $\lambda \in [0, 1]$,

$$\mathcal{E}(\lambda Q^1 + (1 - \lambda)Q^2 | F_t) \geq \lambda \mathcal{E}(Q^1 | F_t) + (1 - \lambda)\mathcal{E}(Q^2 | F_t)$$
Nonlinear Expectations

There is a relation between nonlinear expectations and convex risk measures:

- If (1)-(6) are satisfied, then for each $t$,

$$\rho_t(X) := -\mathcal{E}(X|\mathcal{F}_t)$$

defines a dynamic convex risk measure. These risk measures are time consistent.

- For simplicity, this presentation will discuss nonlinear expectations.

- How could we construct such a family of operators?
In this presentation, we shall consider discrete time processes satisfying ‘Backward Stochastic Difference Equations’.

These are the natural extension of Backward Stochastic Differential Equations in continuous time.

We shall see that every nonlinear expectation satisfying Axioms (1-5) solves a BSDE with certain properties, and conversely.

We also establish necessary and sufficient conditions for concavity (Axiom 6).

To do this, we first need to set up our probability space.
A probabilistic setting

- Let $X$ be a discrete time, finite state process. Without loss of generality, $X$ takes values from the unit vectors in $\mathbb{R}^N$.
- Let $\{\mathcal{F}_t\}$ be the filtration generated by $X$, that is $\mathcal{F}_t$ consists of every event that can be known from watching $X$ up to time $t$.
- Let $M_t = X_t - E[X_t|\mathcal{F}_{t-1}]$. Then $M$ is a martingale difference process, that is $E[M_t|\mathcal{F}_{t-1}] = 0 \in \mathbb{R}^N$. 
Discrete BSDEs (‘D=Difference’)  

A BSDE is an equation of the form:

\[ Y_t - \sum_{t \leq u < T} F(\omega, u, Y_u, Z_u) + \sum_{t \leq u < T} Z_u M_{u+1} = Q \]

- \( Q \) is the terminal condition (in \( \mathbb{R} \)).
- \( F \) is a (stochastic) ‘driver’ function, with \( F(\omega, u, \cdot, \cdot) \) known at time \( u \).
- A solution is an adapted pair \( (Y, Z) \) of processes, \( Y_t \in \mathbb{R} \) and \( Z_t \in \mathbb{R}^{N \times 1} \).
- All quantities are assumed to be \( \mathbb{P} \)-a.s. finite.
Discrete BSDEs (‘D=Difference’)

Equivalently, we can write this in a differenced form:

\[ Y_t - F(\omega, t, Y_t, Z_t) + Z_t M_{t+1} = Y_{t+1} \]

with terminal condition

\[ Y_T = Q. \]

The important detail is that

- The *terminal* condition is fixed, and the dynamics are given in reverse.
- The solution \((Y, Z)\) is adapted, that is, at time \(t\) it depends only on what has happened up to time \(t\).
Suppose we have a market with two assets: a stock $Y$ following a simple binomial price process, and a risk free Bond $B$.

Let $r_t$ denote the one-step interest rate at time $t$.

From each time $t$, there are two possible states for the stock price the following day, $Y(t + 1, \uparrow)$ and $Y(t + 1, \downarrow)$.

Suppose these two states occur with (real world) probabilities $p$ and $1 - p$ respectively.
It is easy to show that there exists a unique ‘no-arbitrage’ price

\[ Y(t) = \frac{1}{1 + r_t} \left[ \pi Y(t + 1, \uparrow) + (1 - \pi) Y(t + 1, \downarrow) \right] \]

\[ = \frac{1}{1 + r_t} E_{\pi} [Y(t + 1)|\mathcal{F}_t]. \]

Here \( \pi \) the ‘risk-neutral probability’, that is, the price today is the average discounted price tomorrow; when \( \pi \) is the probability of a price increase.
Writing $Y_t = Y(t)$ etc... we also know,

$$Y_{t+1} = E_p(Y_{t+1}|\mathcal{F}_t) + L_{t+1}$$

where $E_p(Y_{t+1}|\mathcal{F}_t)$ is the (real-world) conditional mean value of $Y_{t+1}$, and $L_{t+1}$ is a random variable with conditional mean value zero

$$(L_{t+1} = Y_{t+1} - E_p(Y_{t+1}|\mathcal{F}_t)).$$
In the notation we established before, we can define a martingale difference process $M$

$$M_{t+1}(\uparrow) = \begin{bmatrix} 1 - p \\ p - 1 \end{bmatrix}, \quad M_{t+1}(\downarrow) = \begin{bmatrix} -p \\ p \end{bmatrix}. $$

And it is easy to show that $L_{t+1}$ can be written as $Z_t M_{t+1}$, for some row vector $Z_t$ known at time $t$. (Doob-Dynkin Lemma)
We can then do some basic algebra:

\[ Y_{t+1} = E_p(Y_{t+1} | \mathcal{F}_t) + L_{t+1} \]

\[ = Y_t + r_t Y_t - (1 + r_t) Y_t + E_p(Y_{t+1} | \mathcal{F}_t) + Z_t M_{t+1} \]

\[ = Y_t + r_t Y_t - (1 + r_t) \frac{1}{1 + r_t} E_{\pi}(Y_{t+1} | \mathcal{F}_t) + E_p(Y_{t+1} | \mathcal{F}_t) + Z_t M_{t+1} \]

\[ = Y_t + r_t Y_t - E_{\pi}(Y_{t+1} - E_p(Y_{t+1}) | \mathcal{F}_t) + Z_t M_{t+1} \]

\[ = Y_t + r_t Y_t - E_{\pi}(L_{t+1} | \mathcal{F}_t) + Z_t M_{t+1} \]

\[ = Y_t - \left[ - r_t Y_t + Z_t E_{\pi}(M_{t+1} | \mathcal{F}_t) \right] + Z_t M_{t+1} \]

\[ = Y_t - F(Y_t, Z_t) + Z_t M_{t+1} \]
So our one-step pricing formula is equivalent to the equation

\[ Y_{t+1} = Y_t - F(Y_t, Z_t) + Z_t M_{t+1} \]

where

\[ F(Y_t, Z_t) = -r_t Y_t + Z_t E_\pi(M_{t+1}|\mathcal{F}_t) \]

\[ = -r_t Y_t + Z_t \begin{bmatrix} \pi - p \\ p - \pi \end{bmatrix} \]

This is a special case of a BSDE.
Before giving general existence properties of BSDEs, we need the following.

**Definition**
If $Z_t^1 M_{t+1} = Z_t^2 M_{t+1}$ $\mathbb{P}$-a.s. for all $t$, then we write $Z^1 \sim_M Z^2$.
Note this is an equivalence relation for $Z_t \in \mathbb{R}^{N \times 1}$.

**Theorem**
For any $\mathcal{F}_{t+1}$-measurable random variable $W \in \mathbb{R}$ with $E[W|\mathcal{F}_t] = 0$, there exists a $\mathcal{F}_t$-measurable $Z_t \in \mathbb{R}^{N \times 1}$ with

$$W = Z_t M_{t+1}.$$
An Existence Theorem

Theorem
Suppose

(i) \( F(\omega, t, Y_t, Z_t) \) is invariant under equivalence \( \sim_M \)
(ii) For all \( Z_t \), the map

\[
Y_t \mapsto Y_t - F(\omega, t, Y_t, Z_t)
\]

is a bijection

Then a BSDE with driver \( F \) has a unique solution in \( L^1 \).

Corollary

These conditions are necessary and sufficient.
Proof:

Let $Z_t \in \mathbb{R}^{N \times 1}$ solve

$$Z_t M_{t+1} = Y_{t+1} - E[Y_{t+1} | \mathcal{F}_t].$$

Then let $Y_t \in \mathbb{R}$ solve

$$Y_t - F(\omega, t, Y_t, Z_t) = E[Y_{t+1} | \mathcal{F}_t]$$

for the above value of $Z_t$.

Then $(Y_t, Z_t)$ solves the one step equation

$$Y_t - F(\omega, t, Y_t, Z_t) + Z_t M_{t+1} = Y_{t+1},$$

and the result follows by backwards induction.
We wish to ensure that, when $Q^1 \geq Q^2$, the corresponding values $Y_t^1 \geq Y_t^2$ for all $t$.

This will, (eventually), allow us to define a nonlinear expectation $\mathcal{E}$ and obtain the monotonicity and concavity assumptions.

The key theorem here is the *Comparison Theorem*.

**Definition**

We define $\mathcal{J}_t$, the set of possible jumps of $X$ from time $t$ to time $t+1$, by

$$\mathcal{J}_t := \{ i : \mathbb{P}(X_{t+1} = e_i | \mathcal{F}_t) > 0 \}.$$
Comparison Theorem

Theorem

Consider two BSDEs with drivers $F^1$, $F^2$, terminal values $Q^1$, $Q^2$, etc... Suppose that, $\mathbb{P}$-a.s. for all $t$,

(i) $Q^1 \geq Q^2$

(ii) $F^1(\omega, t, Y^2_t, Z^2_t) \geq F^2(\omega, t, Y^2_t, Z^2_t)$

(iii) $F^1(\omega, t, Y^1_t, Z^1_t) - F^1(\omega, t, Y^2_t, Z^2_t)$

$\geq \min_{j\in J} \{(Z^1_t - Z^2_t)(e_j - E[X_{t+1}|\mathcal{F}_t])\}.$

(iv) The map $Y_t \mapsto Y_t - F(\omega, t, Y_t, Z^1_t)$ is strictly increasing in $Y_t$.

Then $Y^1_t \geq Y^2_t$ $\mathbb{P}$-a.s. for all $t$. 
Proof:

Assume $Y_{t+1}^1 - Y_{t+1}^2 \geq 0$, then, omitting $\omega$ and $t$,

$$Y_t^1 - Y_t^2 - F^1(Y_t^1, Z_t^1) + F^1(Y_t^2, Z_t^1)$$
$$- [F^1(Y_t^2, Z_t^1) - F^1(Y_t^2, Z_t^2)] + (Z_t^1 - Z_t^2)M_{t+1}$$
$$= [Y_{t+1}^1 - Y_{t+1}^2] + [F^1(Y_t^2, Z_t^2) - F^2(Y_t^2, Z_t^2)]$$
$$\geq 0.$$  

This must hold $\mathbb{P}$-a.s., so it holds under taking the $\mathcal{F}_t$-conditional essential minimum of all terms. Hence

$$Y_t^1 - Y_t^2 - F^1(Y_t^1, Z_t^1) + F^1(Y_t^2, Z_t^1) \geq 0$$

and then as $Y_t \mapsto Y_t - F(Y_t, Z_t^1)$ is strictly increasing, the result follows by induction.
Given this theory, we can now construct explicit examples of nonlinear expectations.
In fact, every nonlinear expectation can be constructed this way.
BSDEs and Nonlinear Expectations

Theorem

The following statements are equivalent:

(i) $\mathcal{E}(\cdot | \mathcal{F}_t)$ is an $\mathcal{F}_t$-consistent, translation invariant nonlinear expectation. (Axioms 1-5)

(ii) There is an $F$ such that $Y_t = \mathcal{E}(Q | \mathcal{F}_t)$ solves a BSDE with driver $F$ and terminal condition $Q$, where $F$ satisfies the conditions of the comparison theorem, is independent of $Y_t$, and $F(\omega, t, Y_t, 0) = 0 \ P$-a.s. for all $t$.

In this case,

$$F(\omega, t, Y_t, Z_t) = \mathcal{E}(Z_t M_{t+1} | \mathcal{F}_t).$$
Corollary

The nonlinear expectation $\mathcal{E}(\cdot | \mathcal{F}_t)$ has property ‘...’ if and only if $F$ has property ‘...’ (in $Z$), where ‘...’ is any of:

- Concavity
- Positive homogeneity
- Linearity
- Invariance under addition of martingale terms orthogonal to a given process
- (Lipshitz) continuity (in $L^1$ norm)
- etc...

Note, these statements are trivial, given the equivalence

$$F(\omega, t, Y_t, Z_t) = \mathcal{E}(Z_t M_{t+1} | \mathcal{F}_t).$$
The proof of this is simple, but long.

This result holds for both scalar and vector valued nonlinear expectations.

Similar results have been obtained for the scalar Brownian Case, (Coquet et al, 2002), (Hu et al, 2008).

In discrete time everything is simpler, and one can even obtain similar results for the more general nonlinear evaluations (Cohen & Elliott, forthcoming)
An Example

To demonstrate the complexity that can be achieved, consider a two step world where $X_t$ takes one of two values with equal probability.

Assume that $Z_t$ is written in the form $Z_t = [z, -z]$, which is unique up to equivalence $\sim_{M_t}$. We consider the concave function

$$F(\omega, t, Y_t, Z_t) = \min_{\pi \in [0.1, 0.9]} \left\{ 2(\pi - 0.5)z + \gamma(\pi - 0.5)^2 \right\},$$

where $\gamma$ is a ‘risk aversion’ parameter (the smaller the value of $\gamma$, the more risk averse), which we shall set to $\gamma = 10$. 
Other Results:

- One can show under what conditions a generic monotone map $L^2(\mathcal{F}_T) \to \mathbb{R}$ can be extended to an $\mathcal{F}_t$ consistent nonlinear expectation.

- It is also possible, in general, to determine under what conditions the driver $F$ can be determined from the solutions $Y_t$, even when the comparison theorem and normalisation conditions do not hold.

- These results are significantly stronger than available in continuous time.
Conclusions

- The theory of BSDEs can be expressed in discrete time.
- Various continuous time results, such as the comparison theorem, extend naturally to the discrete setting.
- The discrete time proofs are often simpler than in continuous time, and give stronger results.
- It forms a natural setting for nonlinear expectations, as every nonlinear expectation solves a BSDE.
- This has various implications for problems in economic regulation, and in other areas of optimal stochastic control.