Multi-Stock Portfolio Optimization under Prospect Theory

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Summary

- We analyzes how a cumulative prospect theory agent optimizes her portfolio.
- Whereas this has been solved in a one-period setting with one riskless and one risky asset, we consider \( n \) risky assets.
- Our main result is a two-fund separation between the riskless bond and a mean-variance-portfolio. All CPT-agents invest into the same mean-variance-portfolio, independent of their individual risk preferences.
- Further we derive this mean-variance-portfolio explicitly.
- The optimal portfolio is thus provided explicitly up to solving a one-dimensional problem. Thus we escape the curse of dimensionality.
Agenda

Literature

The Setting

Optimal Portfolios

Specific Value Functions

Numerical Illustration

Conclusion
Related Literature

- Separation results have a long history: Tobin (1958), Owen and Rabinovitz (1983), Chamberlain (1983), ...
- Solved the optimization with one risky asset: Bernard and Ghossoub (2010), He and Zhou (2011).
Financial Market

- One-period Financial Market: one bond with safe return $r$, $n$ risky assets with excess return $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$.
- Wealth at the end of the period is

$$W = W_0(1 + r) + \pi \cdot \bar{x},$$

where $W_0$ is endowment and $\pi$ vector of invested amounts.
Cumulative Prospect Theory

The three key elements of CPT of Tversky and Kahnemann (1992)

1 Reference point

\[ W^{\text{ref}} := a \cdot \bar{x} + b \cdot \pi + p, \]

for arbitrary vectors \( a, b \in \mathbb{R}^n \), and scalar \( p \). This definition includes risk-free return \( W_0(1 + r) \), expected wealth \( EW \), and stock indices. We consider investment in excess of benchmark portfolio \( a \)

\[ \zeta := \pi - a, \]

and we further simplify by setting \( y := \bar{x} - b \) and \( c := p + a \cdot b - W_0(1 + r) \), which yields for the deviation from the reference point

\[ W - W^{\text{ref}} = \zeta \cdot y - c =: D(\zeta). \]
Cumulative Prospect Theory II

The three key elements of CPT

2 ‘S-shaped’ value function

\[ u(x) := \begin{cases} 
  u^+(x) & \text{for } x \geq 0 \\
  -u^-(x) & \text{for } x < 0 
\end{cases} \]

where \( u^+ : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) and \( u^- : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) are differentiable, increasing and convex functions.

3 Probability distortions \( T^+ : [0, 1] \rightarrow [0, 1] \) for gains and \( T^- : [0, 1] \rightarrow [0, 1] \) for losses, which increase small probabilities and shrink large probabilities.
Overall prospect value is defined by

\[ V(D(\zeta)) := V^+(D(\zeta)) - \lambda V^-(D(\zeta)), \]

where

\[ V^+(D(\zeta)) := \int_0^\infty u^+(x) \, d[-T^+(1 - F(x))] , \]

\[ V^-(D(\zeta)) := \int_{-\infty}^0 u^-(x) \, d[T^-(F(x))] . \]

**Assumption**

The prospect values of gains \( V^+(D(\zeta)) \) and losses \( V^-(D(\zeta)) \) are finite for all \( \zeta \in \mathbb{R}^n \).
Distribution of Returns

Assumption

The random vector $y$ of excess returns on the risky assets has an elliptically symmetric distribution.

- For mean vector $\mu$, positive definite covariance matrix $\Sigma$, and shape $g$ we write
  $$y \sim EC_n(\mu, \Sigma; g).$$

We use the fact that

$$D(\zeta) = \zeta \cdot y - c \sim EC_1(\bar{\mu}, \bar{\sigma}; g)$$

where

$$\bar{\mu} := \zeta \cdot \mu - c, \quad \bar{\sigma}^2 := \zeta^\top \Sigma \zeta.$$
Distribution of Returns II

- Elliptical distributions are central in portfolio optimization.
- Our results do probably not extend beyond elliptical returns.
- Fortunately, elliptical distributions include the normal distribution, the Cauchy distribution, the $t$ distribution, the exponential power family, the logistic family, normal-variance mixture distributions, symmetric stable distributions, the symmetric generalized hyperbolic distribution, ...
- As found in numerous empirical studies, many elliptical distributions provide an accurate fit to stock data, in particular the $t$ distribution.
CPT Selection Problem

We solve the optimization problem of maximizing the prospect value
\[
\sup_{\zeta \in \mathbb{R}^n} V(D(\zeta)).
\]

The mean-variance-portfolio is defined by
\[
\zeta_M := \Sigma^{-1}\mu,
\]
and the optimal participation level in the mean-variance-portfolio is
\[
k^* := \arg\max_{k \in \mathbb{R}} V(D(k\zeta_M)).
\]
Unconstrained Case

**Assumption**

There exists $\zeta^* \in \mathbb{R}^n$ with $\zeta^* = \arg \max_{\zeta \in \mathbb{R}^n} V(D(\zeta))$.

**Theorem**

The optimal portfolio of the CPT-agent is given by

$$
\zeta^* = k^* \zeta_M = k^* \Sigma^{-1} \mu.
$$

- The theorem establishes the two-fund separation for well-posed problems and derives the mean-variance-portfolio explicitly.
- The $n$-dimensional portfolio optimization problem is reduced to the one-dimensional optimization of $k^*$. This is appealing for traders who optimize high-dimensional portfolios.
Constrained Case

- We drop the well-posedness assumption.
- In an ill-posed problem the maximal prospect value is not gained by a finite portfolio. It sets wrong incentives as the agent strives for an infinite portfolio, since the trade-off between gains and losses is not present.
- He and Zhou (2011) state the cut-off between well-posed and ill-posed models.
- The optimal infinite portfolio would be exposed to infinite risk. Since this is not favorable, our approach is to restrict the risk by an external constraint, which ensures well-posedness.
Constrained Case II

- We allow for any risk measure $\rho = \rho(D(\zeta)) = \rho(\bar{\mu}(\zeta), \bar{\sigma}(\zeta))$, with derivatives $\rho_{\bar{\mu}} < 0$ and $\rho_{\bar{\sigma}} > 0$, and

$$\lim_{|\zeta| \to \infty} \rho(\bar{\mu}(\zeta), \bar{\sigma}(\zeta)) = \infty.$$ 

- Examples include \textit{VaR} and \textit{AVaR} for sufficiently small confidence levels.

- We constrain the optimization problem to the compact set of admissible portfolios

$$K := \{ \zeta \in \mathbb{R}^n : \rho(D(\zeta)) \leq M \},$$

and the two-fund separation can also be derived.
Common Choices of Value Functions

- We study the piecewise power value function
  - $u^+(x) := x^\alpha$, where $0 < \alpha < 1$ and $x \geq 0$,
  - $u^-(x) := x^\beta$, where $0 < \beta < 1$ and $x > 0$.

**Proposition**

For $\alpha < \beta$ the optimal portfolio of the CPT-agent is given by

$$\zeta^* = k^* \zeta_M = \begin{cases} k_1 \zeta_M, & \text{if } V(D(k_1 \zeta_M)) \geq V(D(k_2 \zeta_M)), \\ k_2 \zeta_M, & \text{if } V(D(k_1 \zeta_M)) < V(D(k_2 \zeta_M)), \end{cases}$$

where $k_1$ and $k_2$ have closed forms.
We consider the piecewise exponential value function

- $u^+(x) := \frac{1}{\nu}(1 - e^{-\nu x})$, where $0 < \nu$ and $x \geq 0$,
- $u^-(x) := \frac{1}{\xi}(1 - e^{-\xi x})$, where $0 < \xi$ and $x > 0$,

**Proposition**

*If it holds*

$$\lambda > \frac{\xi T^+(1 - \Theta(-d))}{\nu T^-(\Theta(-d))},$$

*then two-fund separation holds.*
Numerical Illustration

- We consider the portfolio of five major US stocks and their daily returns of Hu and Kercheval (2010) and derive the mean-variance portfolio by

\[ \zeta_M = [102.54, 390.1, -433.05, 81.91, -41.64]. \]

- After specifying a piecewise power CPT agent, we derive the total amount invested in the five risky stocks is

\[ R := \sum_{i=0}^{n} \zeta_i^* = \sum_{i=0}^{n} k^* \zeta_{M,i}, \]

and plot it as a function of power utility exponents \( \alpha \) and \( \beta \).
Multivariate Student Stock Returns

amount invested in the stocks

\( \alpha \)

\( \beta \)
Multivariate Normal Stock Returns

amount invested in the stocks

\( \nu \)

\( \xi \)
Conclusion

- We solve the portfolio optimization under CPT for multiple risky stocks, and we establish a two-fund separation between the bond and a mean-variance portfolio.
- Our CPT setting is fairly general. However we restrict to elliptical asset returns.
- We also solve ill-posed problems by imposing a risk constraint.
- We analyze the piecewise power and exponential specification.
- Overall we escape the curse of dimensionality, and the semi-closed form simplifies further analytical and numerical analysis.
Proposition

For $\alpha < \beta$ the optimal portfolio of the CPT-agent is given by

$$\zeta^* = k^* \zeta_M = \begin{cases} k_1 \zeta_M, & \text{if } V(D(k_1 \zeta_M)) \geq V(D(k_2 \zeta_M)), \\ k_2 \zeta_M, & \text{if } V(D(k_1 \zeta_M)) < V(D(k_2 \zeta_M)), \end{cases}$$  \hspace{1cm} (1)

where

$$k_1 = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta - \alpha}} \left( \frac{\int_0^\infty x^\alpha f^+ \left( \frac{x}{\sigma_M} - \sigma_M \right) \, dx}{\lambda \int_{-\infty}^0 (-x)^\beta f^- \left( \frac{x}{\sigma_M} - \sigma_M \right) \, dx} \right)^{\frac{1}{\beta - \alpha}} < \infty, \hspace{1cm} (2)$$

and

$$k_2 = -\left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta - \alpha}} \left( \frac{\int_0^\infty x^\alpha f^+ \left( -\frac{x}{\sigma_M} - \sigma_M \right) \, dx}{\lambda \int_{-\infty}^0 (-x)^\beta f^- \left( -\frac{x}{\sigma_M} - \sigma_M \right) \, dx} \right)^{\frac{1}{\beta - \alpha}} > -\infty. \hspace{1cm} (3)$$

Thus optimal total investment is given by $\pi^* = k^* \zeta_M + a$. 