A General Analytical Approach for Drawdown (Drawup) Risks of Time-Homogeneous Markov Models

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1 Aim of the research

Drawdowns relate to an investor's sustained loss from a market peak. It is one of the most frequently quoted indices for downside risks by mutual funds and commodity trading advisers through performance measures such as the Calmar ratio, the Sterling ratio, the Burke ratio, etc. In finance and risk management, a significant body of literature has been developed on assessing, managing, controlling, and reducing drawdown risks. In addition, it is interesting that various a priori unrelated problems in other areas are closely connected to the drawdown problematic. For instance, the pricing of Russian options, optimal dividend payment strategies, change point detection which is used to test whether or not a change in distribution of a time series has occurred.

A lot of analytical approaches have been introduced and developed to study drawdown risks for various structure models. Ito's excursion theory is the major approach for spectrally negative Lévy processes; see e.g., Avram et al. [2] and Mijatovic and Pistorius [5]. However, the derivation of this approach is usually very involved and it is difficult to be applied to other classes of Markov processes. On the other hand, Lehoczky [4] used an approximation technique for time-homogeneous diffusion processes. Later on, this approach was generalized under the same framework In comparison to excursion theory, the approximation approach is more general and the derivation is also more intuitive and simple. However, it is not valid for Markov processes with upside jumps. There are some other techniques in the literature such as martingale theory (Asmussen et al. [1]) and occupation density (Ivanovs and Palmowski [3]).

In this paper, we propose a new analytical technique to study drawdown (drawup) risks for time-homogeneous Markov models. Our approach can be applied to a large class of Markov processes: Lévy process with two-sided jumps, spectrally negative diffusion processes, Markov additive processes, etc. The idea and derivation of our approach is very simple and the key step is to bound drawdown quantities using first passage time quantities. Essentially, our approach shows analytically a connection between drawdown risks and default risks.

2 Principal results

We only consider drawdown risks and all the results below can be parallelly extended to drawup, which is the dual of drawdown measuring the increase in value from the historical trough over a given period of time. We consider a time-homogeneous Markov processes $X = \{X_t, t \geq 0\}$ defined on a filtered probability space satisfying the usual conditions. We define the first passage times of

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X above and below $x \in \mathbb{R}$ by

$$T_x^+ = \inf\{t \ge 0 : X_t > x\} \quad \text{and} \quad T_x^- = \inf\{t \ge 0 : X_t < x\}.$$

The drawdown process of X is defined as $Y = M - X$, where $M_t = \sup_{0\le u\le t} X_t$ is the running maximum of X up to time t. For a pre-specified drawdown size $a > 0$, we denote by

$$\tau_a = \inf\{t \ge 0 : Y_t > a\}$$

the first time that Y exceeds the threshold a. Given that $(\max_{0\le s\le t} Y_s \le a) = (\tau_a > t)$ almost surely, the distributional study of the maximum drawdown is equivalent to the study of the stopping time τ_a. For ease of notation, for $u \le x \le v$, we define these two functions which relate to the two-sided exit problem of X by

$$B^{(q)}(x; u, v) = \mathbb{E}_x\left[e^{-qT_v^+} 1_{\{T_v^+ < T_u^-\}}\right] \text{ and } C^{(q,s)}(x; u, v) = \mathbb{E}_x\left[e^{-qT_u^- - s(u - X_{T_u^-})} 1_{\{T_u^- < T_v^+\}}\right]$$

Assumption 2.1 *For $q, s \ge 0$ and $u < x$, the boundary derivatives*

$$B_v^{(q)}(x; u, x) := \lim_{\varepsilon\to 0+} (B^{(q)}(x; u, x+\varepsilon) - 1)/\varepsilon \text{ and } C_v^{(q,s)}(x; u, x) := \lim_{\varepsilon\to 0+} C^{(q,s)}(x; u, x+\varepsilon)/\varepsilon,$$

exist and are continuous w.r.t. u.

Theorem 2.1 *Consider a stationary time-homogeneous Markov process X satisfying Assumption 2.1. For $q, s \ge 0$ and $a > 0$, we have*

$$\mathbb{E}\left[e^{-q\tau_a - sY_{\tau_a}}\right] = e^{-sa}\frac{C_v^{(q,s)}(0; -a, 0)}{-B_v^{(q)}(0; -a, 0)}.$$

Theorem 2.2 *Consider a spectrally negative time-homogeneous Markov process X satisfying Assumption 2.1. Suppose that $\mathbb{E}_x\left[e^{-q\tau_a - sY_{\tau_a} - r(M_{\tau_a} - x_0)}\right]$ is differentiable w.r.t. x. For $q, r, s \ge 0$, we have*

$$\mathbb{E}_x\left[e^{-q\tau_a - sY_{\tau_a} - r(M_{\tau_a} - x)}\right] = e^{-sa}\int_x^\infty \exp\left\{-\int_x^y \left(r - B_v^{(q)}(z; z-a, z)\right)\mathrm{d}z\right\} C_v^{(q,s)}(y; y-a, y)\mathrm{d}y.$$

Remark 2.1 *Examples of Markov processes which are analytically tractable for the two-sided exit problems include spectrally Lévy processes, jump diffusion processes with phase-type distributed jumps, meromorphic Lévy processes, and spectrally negative Markov additive processes. By our results, we immediately obtain drawdown formulas for these classes of Markov models.*

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