Discrete-time Semi-Markov Random Evolutions: Theory and Applications in Finance

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Abstract

This talk presents a new model for a stock price in the form of a geometric Markov renewal process (GMRP) which is one of many examples of discrete-time semi-Markov random evolutions (DTSMRE). We study asymptotic properties of the DTSMREs, namely, averaging, diffusion approximation and normal deviations by martingale weak convergence method. We also consider controlled DTSMREs and find Hamilton-Jacobi-Bellman equation for them. As applications we present European call option pricing formula for GMRP and find optimal cost function for the controlled GMRP.

Discrete-time Semi-Markov Random Evolutions. Let $\mathbb{N}$ be the set of non-negative integer numbers, $\mathbb{R}_+ := [0, \infty)$, and $\mathcal{B}_+$ be the Borel sets of $\mathbb{R}_+$. Let $(E, \mathcal{E})$ be a measurable space with countably generated σ-algebra and $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbf{P})$ be a stochastic basis on which we consider a Markov renewal process $(x_n, \tau_n, n \in \mathbb{N})$ in discrete time $k \in \mathbb{N}$, with state space $(E \times \mathbb{R}_+, \mathcal{E} \otimes \mathcal{B}_+)$. Let $q(x, B, k) := \mathbf{P}(x_{n+1} \in B, \tau_{n+1} - \tau_n = k \mid x_n = x)$, for $x \in E$, $B \in \mathcal{E}$, and $k, n \in \mathbb{N}$, the semi-Markov kernel. The process (x_n) is the embedded Markov chain of the semi-Markov chain, with transition kernel $P(x, dy)$ given by $P(x, B) := q(x, B, \mathbb{N})$. The semi-Markov kernel q is written as $q(x, dy, k) = P(x, dy)f_{xy}(k)$, where $f_{xy}(k) := \mathbf{P}(\tau_{n+1} - \tau_n = k \mid x_n = x, x_{n+1} = y)$, the conditional distribution of the sojourn time in state x given that the next visited state is y. Define also the counting process of jumps $\nu_k = \max\{n : \tau_n \leq k\}$, and the discrete-time semi-Markov chain z_k by $z_k = x_{\nu_k}$, for $k \in \mathbb{N}$. Define now the backward recurrence time process $\gamma_k := k - \tau_{\nu_k}$, $k \geq 0$, and the filtration $\mathcal{F}_k := \sigma(z_\ell, \gamma_\ell; \ell \leq k)$, $k \geq 0$. The process $(z_k, \gamma_k), k \geq 0$, is a Markov chain. Let us consider a separable Banach space $\mathbb{B}$ of real-valued measurable functions defined on $E \times \mathbb{N}$, endowed with the sup norm $\|\cdot\|$ and denote by $\mathcal{B}$ its Borel σ-algebra. Let $P^\sharp$ be the transition operator kernel of the Markov chain $(z_k, \gamma_k), k \geq 0$, and its stationary distribution $\pi^\sharp(dx \times \{k\})$. The probability measure π defined by $\pi(B) := \pi^\sharp(B \times \mathbb{N})$ is the stationary probability of the SMC (z_k). Let Π be the stationary projection operator on the null space of the (discrete) generating operator $Q^\sharp := P^\sharp - I$, The potential operator of $Q^\sharp$, denoted by R_0 (see

[3]). Let us given a family of linear bounded contraction operators $D(x), x \in E$, (i.e. $\|D(x)\varphi\| \le \|\varphi\|$), defined on $\mathbb{B}$, where the maps $D(x)\varphi : E \to \mathbb{B}$ are $\mathcal{E}$-measurable, $\varphi \in \mathbb{B}$. Denote by I the identity operator on $\mathbb{B}$. Let $\Pi\mathbb{B} = \mathcal{N}(Q^\sharp)$ be the null space, and $(I - \Pi)\mathbb{B} = \mathcal{R}(Q^\sharp)$ be the range values space of operator $Q^\sharp$. We will suppose here that the Markov chain $(z_k, \gamma_k, k \in \mathbb{N})$ is uniformly ergodic, that is, $\sup_{\|\varphi\|\le 1} \left\|((P^\sharp)^n - \Pi)\varphi\right\| \to 0$, as $n \to \infty$, for any $\varphi \in \mathbb{B}$. Notice that this condition implies the exponential ergodicity of the Markov chain. In that case, the transition operator is reducible-invertible on $\mathbb{B}$. Thus, we have $\mathbb{B} = \mathcal{N}(Q^\sharp) \oplus \mathcal{R}(Q^\sharp)$, the direct sum of the two subspaces. The domain of an operator A on $\mathbb{B}$ is denoted by $\mathcal{D}(A) := \{\varphi \in \mathbb{B} : A\varphi \in \mathbb{B}\}$.

A (forward) discrete-time semi-Markov random evolution (DTSMRE) $\Phi_k, k \in \mathbb{N}$, on $\mathbb{B}$, is defined by

$$\Phi_k\varphi = D(z_k)D(z_{k-1})\cdots D(z_2)D(z_1)\varphi, \quad k \ge 1, \text{ and } \Phi_0 = I.$$

for any $\varphi \in \mathbb{B}_0 := \cap_{x\in E}\mathcal{D}(D(x))$. Thus we have $\Phi_k = D(z_k)\Phi_{k-1}$.

Applications in Finance. As one of many applications of DTSMREs in finance and insurance, we consider here the geometric Markov renewal process (GMRP) (see [4]) $S_n := S_0 \prod_{i=1}^{\nu_n}(1 + \rho(x_i))$, which models a stock price with many possible future values. Function $\rho(x)$ is continuous and bounded on E. The GMRP is a generalization of Cox-Ross-Rubinstein binomial model (see [2]) and Aase geometric compound Poisson model (see [1]). We consider this process in series scheme $S_t^\epsilon := \prod_{i=1}^{\nu_{t/\epsilon}}(1 + \epsilon\rho(x_i))$ and obtain ergodic, diffusion and normal deviation of the stock prices in this case. We present European call option pricing formulas for the diffusion and normal deviations cases. We also consider controlled GMRP (CGMRP) $S_n := S_0 \prod_{i=1}^{\nu_n}(1 + \rho(x_i, u_i))$, where u_i is a control process (e.g., Markov chain in some control space U). For CGMRP we obtain the optimal cost function by deriving and solving Hamilton-Jacobi-Bellman equation.

Keywords : Finance, semi-Markov random evolution, geometric Markov renewal process (GMRP), controlled GMRP, European option pricing formulas, optimal cost function, HJB equation.

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