Value-at-Risk bounds with variance constraints

C. Bernard, L. Rüschendorf, S. Vanduffel

Key-words: Value-at-Risk, Convex order, Model risk, Rearrangement algorithm.

Extended Abstract

Aim of the research: In this paper, we study bounds for the Value-at-Risk (VaR) of sums of risks with known marginal distributions (describing the stand-alone risks) under the additional constraint that the variance of the sum is known (which partially describes the dependence between risks). This setting is of significant interest, as in many practical situations it corresponds closely to the maximum information at hand when assessing the VaR of a portfolio. For example, in the context of credit risk portfolio models one typically has knowledge about the marginal risks (through so-called PD, EAD and LGD models) and the variance of the aggregate risk (sum of the individual losses) is also often available (as obtained from default correlation models or through statistical analysis of observed credit losses). The same setting also appears in the context of risk aggregation and solvency calculations (Basel III and Solvency II). Banks and insurers usually have models to estimate risk distributions and VaR per risk type (credit risk, market risk, operational risk,. . .) and per business line, and next they rely on a correlation matrix to obtain the VaR of the aggregated portfolio. Taking into account this information we asses how much model risk is left in the computation of VaR.

Bounds on Value-at-Risk in the unconstrained case (only the marginal distributions are fixed) have been studied by Rüschendorf (1982) (in the case of two risks) and more recently by several other authors including Denuit et al. (1999), Embrechts and Puccetti (2006) and Embrechts et al. (2013). In particular, there exist several explicit results on VaR bounds when the marginals are identically distributed. In the inhomogeneous case, however, the analysis is fairly more complicated and explicit results are scarce. Puccetti and Rüschendorf (2012a) propose the rearrangement algorithm (RA) as a practical way to approximate bounds of the portfolio Value-at-Risk; see also Embrechts et al. (2013). So far, numerical experiments have shown that the RA presents very good accuracy.

There are already results regarding VaR bounds in the presence of partial information on the dependence of the underlying risks in the sum; see e.g. Rüschendorf (1991) and Embrechts et al. (2013) for results when some of the bivariate distributions are known. However, the bounds that are proposed in these papers are often hard to compute numerically, especially for higher dimensions and in the inhomogeneous case. Moreover, in practice, bivariate distributions are usually not known. In contrast, the variance of the entire portfolio sum can often be statistically estimated with sufficient degree of accuracy or its value can be implied by the availability of the correlations between risks. Intuitively, as the variance measures the average spread of the aggregate portfolio loss around the mean, one could expect that its knowledge has a significant impact on the maximum possible VaR. Hence, in this paper we study bounds on the Value-at-Risk of a sum with a known maximum variance and with fixed marginals.

Principal results: We make several contributions in this paper. First, we give simple upper and lower VaR bounds in terms of the tail Value-at-Risk in the unconstrained case when only marginal information is available. These unconstrained bounds are also valid for the case of heterogeneous portfolios and several examples illustrate that they turn out to be reasonably sharp. We find that the upper and lower bounds are sharp if it is possible to construct random variables, which are mixing on the upper part resp. on the lower part of the distribution, i.e., when their sum is constant. We also establish a connection between the problem of obtaining good VaR bounds and results on convex ordering, which leads to further improved bounds in certain cases.

Next, we consider an additional constraint on the variance of the joint portfolio and give an analytical bound in this case. We show that these (constrained) VaR bounds can be significantly tighter than the bounds in the unconstrained case. We describe conditions to ensure sharpness of these bounds. To obtain sharp VaR bounds for the constrained case, we show that it is necessary to make the distribution of the aggregate risk as flat as possible in the upper part as well as in the lower part while considering at the same time the variance constraint.

This insight gives the intuition to develop a new algorithm for obtaining approximate sharp bounds in the constrained case. This algorithm extends the rearrangement algorithm, which was proposed to approximate sharp VaR bounds in the unconstrained case by Puccetti and Rüschendorf (2012a) and Embrechts et al. (2013). The basic idea is that we simultaneously rearrange the upper and the lower part of the distribution of the sum and move in some systematic way through the domain of the random sum in order to fulfill the variance constraint. A series of examples show that the extended rearrangement algorithm (ERA) works well. In particular, we see under which conditions the additional variance constraint leads to essentially improved VaR bounds.

Major Conclusions: We criticize the way internal models in regulatory frameworks are used for establishing capital requirements. In particular, we do not recommend using them for computing portfolio VaR at high confidence levels (e.g., 99.5%) as the basis for setting capital requirements. It seems more effective to impose additional restrictions on the internal models used for setting the capital requirements, or even to enforce the use of a single model to this purpose. By doing so, the capital requirements become better comparable across different institutions, which also enhances fair competition.

References

Embrechts, P., Puccetti, G., (2009). Bounds for the sum of dependent risks having overlapping marginals. *Journal of Multivariate Analysis*, **101**, 177–190.

Embrechts, P., Puccetti, G., Rüschendorf, L., (2013). Model uncertainty and VaR aggregation, *Journal of Banking and Finance*, **37**(8), 2750–2764.

Puccetti, G., Rüschendorf, L., (2012a). Computation of sharp bounds on the distribution of a function of dependent risks. *Journal of Computational and Applied Maths*, **236**(7), 1833–1840.

Rüschendorf, L. (1982). Random variables with maximum sums. *Advances in Applied Probability*, **14**(3), 623–632.

Wang, B., Wang, R. (2011). The complete mixability and convex minimization problems for monotone marginal distributions. *Journal of Multivariate Analysis*, **102**, 1344–1360.