Exponential Family Techniques in the Lognormal Left Tail

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[1] with Jens Ledet Jensen & Leonardo Rojas-Nandayapa

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n assets in portfolio, long position

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value after period: $S = \sum_{i=1}^{n} X_i,$

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Only ref's:
Rojas-Nandayapa PhD thesis 2008
Gulisashvili & Tankov 2013

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SA-Rojas-Nandayapa 2008: Gaussian copula, different μ_i, σ_i^2

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Saddlepoint approximation

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Do the same in lognormal left tail

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Took $\mu = 0$ (e^{μ} scaling factor)

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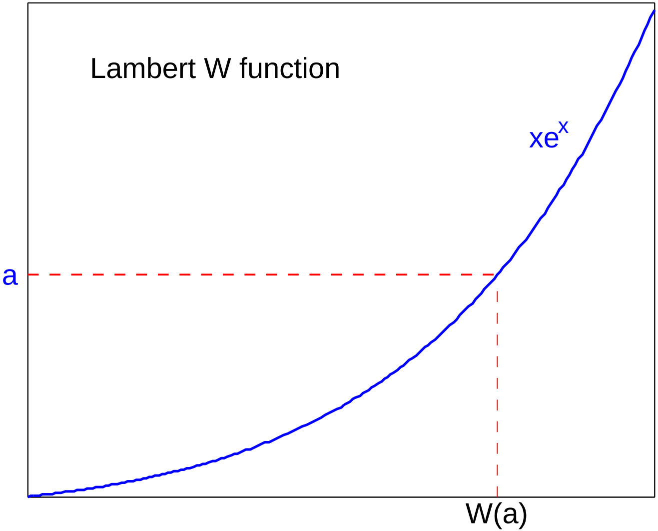
Approach: Laplace method;

Gives asymptotics in terms of Lambert's W


Lambert W function
xe^x
a
W(a)

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Theorem

$$\mathcal{L}(\theta) \sim \frac{\exp\left\{-\dfrac{\mathcal{W}^2(\theta\sigma^2) + 2\,\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\}}{\sqrt{1 + \mathcal{W}(\theta\sigma^2)}}, \quad \theta \to \infty.$$

Simulation algorithm

Variant of approximation:

$$\mathcal{L}(\theta) \; = \; \exp\left\{-\frac{\mathcal{W}^2(\theta\sigma^2) + 2\,\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\}\mathbb{E}g(\theta,\sigma^2,V)\,, \quad V \sim N(0,1)$$

Algorithm estimates $\mathbb{E}g(\theta,\sigma^2,V)$ by MC.

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Write $\kappa(\theta) = \log \mathcal{L}(\theta)$:

$$f_\theta(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\{-\theta x - \log^2 x/2\sigma^2 - \kappa(\theta)\}$$

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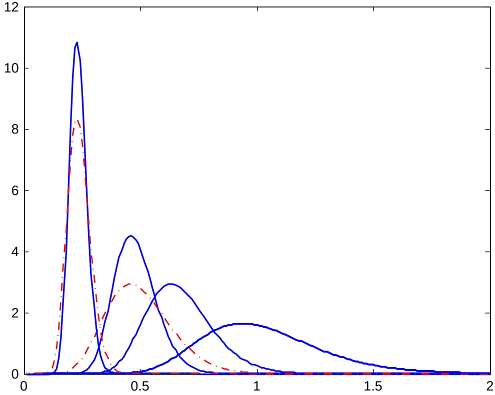

Define $\mu_\theta = -\mathcal{W}(\theta\sigma^2), \qquad \sigma^2_\theta = \dfrac{\sigma^2}{1+\mathcal{W}(\theta\sigma^2)}.$

Then $\lim_{\theta\to\infty} \dfrac{\mathbb{E}_\theta[X]}{\mathrm{e}^{\mu_\theta}} = 1, \qquad \lim_{\theta\to\infty} \dfrac{\mathbb{V}\mathrm{ar}_\theta[X]}{\mathrm{e}^{2\mu_\theta}\left(\mathrm{e}^{\sigma^2_\theta}-1\right)} = 1.$

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3) normal

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Simulation algorithm:

Generate r.v. from F_θ by acceptance-rejection with gamma proposal

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insert approximation for $\mathbb{E}_\theta X = x$ to get approximation
$\widetilde{\theta}(x) = \dfrac{\gamma(x)\mathrm{e}^{\gamma(x)}}{\sigma^2}$
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Simulation algorithm:
Importance sampling, simulate from F_θ and simulate $\mathcal{L}(\theta)$

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Table : Approximation of the CDF of a lognormal sum with $n = 16$ and $\sigma = 0.125$ (period = a quarter).

x	$z = nx$	$\tilde{\theta}(x)$	Saddle	Simulation
0.9000	14.40	7.99	1.63e-04	1.63e-04 ± 1.96e-06
0.9094	14.55	7.18	5.51e-04	5.50e-04 ± 6.32e-06
0.9187	14.70	6.40	1.66e-03	1.66e-03 ± 1.82e-05
0.9281	14.85	5.64	4.50e-03	4.48e-03 ± 4.67e-05
0.9375	15.00	4.90	1.10e-02	1.10e-02 ± 1.08e-04
0.9469	15.15	4.19	2.42e-02	2.40e-02 ± 2.23e-04
0.9563	15.30	3.49	4.85e-02	4.85e-02 ± 4.18e-04

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GT can do dependence, different μ_i, σ_i^2
We are precise in range $\mathbb{P}(S_n \leq z) \in (e-4, e-2)$, GT not.

Thank you !!!