A New Approach to Assessing Model Risk in High Dimensions

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Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of d individual dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of the portfolio?
- A non-parametric method based on the data at hand.
- Analytical expressions for these maximum and minimum

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- A non-parametric method based on the data at hand.
- Analytical expressions for these maximum and minimum
- Implications:
  - Current regulation is subject to very high model risk, even if one knows the multivariate distribution almost completely.
  - Able to quantify model risk for a chosen risk measure. We can identify for which risk measures it is meaningful to develop accurate multivariate models.

Model Risk

1. Goal: Assess the risk of a portfolio sum $S = \sum_{i=1}^{d} X_i$.
2. Choose a risk measure $\rho(\cdot)$, "fit" a multivariate distribution for $(X_1, X_2, ..., X_d)$ and compute $\rho(S)$
3. How about model risk? How wrong can we be?

$$\rho_{\mathcal{F}}^{+} := \sup\left\{\rho\left(\sum_{i=1}^{d} X_i\right)\right\}, \quad \rho_{\mathcal{F}}^{-} := \inf\left\{\rho\left(\sum_{i=1}^{d} X_i\right)\right\}$$

where the supremum and the infimum are taken over all other (joint distributions of) random vectors $(X_1, X_2, ..., X_d)$ that "agree" with the available information

Choice of the risk measure

- **Variance** of X
- **Value-at-Risk** of X at level $p \in (0,1)$

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geqslant p\} \qquad (1)$$

- **Tail Value-at-Risk** or **Expected Shortfall** of X

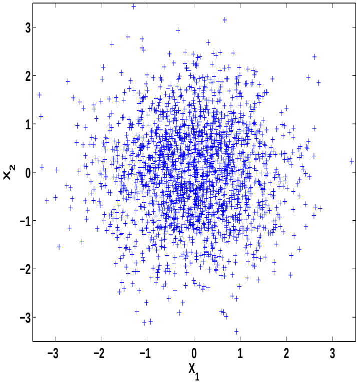
$$\mathrm{TVaR}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_u(X)\mathrm{d}u \qquad p \in (0,1)$$

and $p \to \mathrm{TVaR}_p$ is continuous.

Assessing Model Risk on Dependence with d Risks

- Marginals known
- Dependence unknown
- Already a challenging problem in $d \geqslant 3$ dimensions
  - Wang and Wang (2011, JMVA)
  - Embrechts, Puccetti, Rüschendorf (2013, JBF): algorithm (RA) to find bounds on VaR
- Issues
  - bounds are generally very wide
  - ignore all information on dependence.

Illustration with marginals N(0,1)



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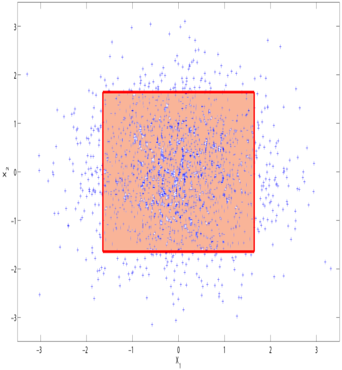

$$\mathcal{F}_1 = \bigcap_{k=1}^{2} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$$

Our assumptions on the cdf of $(X_1, X_2, ..., X_d)$

$\mathcal{F} \subset \mathbb{R}^d$ ("trusted" or "fixed" area)
$\mathcal{U} = \mathbb{R}^d \backslash \mathcal{F}$ ("untrusted").
We assume that we know:

(i) the marginal distribution F_i of X_i on $\mathbb{R}$ for $i = 1, 2, ..., d$,
(ii) the distribution of $(X_1, X_2, ..., X_d) \mid \{(X_1, X_2, ..., X_d) \in \mathcal{F}\}$.
(iii) $P((X_1, X_2, ..., X_d) \in \mathcal{F})$

- joint distribution of $(X_1, X_2, ..., X_d)$ is thus known if $\mathcal{F} = \mathbb{R}^d$ and $\mathcal{U} = \emptyset$.
- When only marginals are known: $\mathcal{U} = \mathbb{R}^d$ and $\mathcal{F} = \emptyset$.
- **Our Goal: Find bounds on $\rho(S) := \rho(X_1 + ... + X_d)$ when $(X_1, ..., X_d)$ satisfy (i), (ii) and (iii).**

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2 methods: **non-parametric approach** or Monte-Carlo simulation from **theoretical bounds**.

First Approach

Approximation of Bounds

(for variance and TVaR)

Example:

$N = 8$ observations, $d = 3$ dimensions
and 3 observations trusted ($\ell_f = 3$, $p_f = 3/8$)

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad S_N = \begin{bmatrix} 8 \\ 3 \\ 5 \\ 3 \\ 8 \\ 4 \\ 4 \\ 9 \end{bmatrix}$$

- The matrix M is split into two parts: $\mathcal{F}_N$: trusted observations, $\mathcal{U}_N$: "untrusted" part.
- Rearranging the values x_{ik} $(i = 1, 2, ..., N)$ within the $k-$th column does not affect the marginal distribution X_k but only changes the observed dependence.
- ℓ_f : number of elements in $\mathcal{F}_N$, ℓ_u : number of elements in $\mathcal{U}_N$

$$N = \ell_f + \ell_u.$$

- M has ℓ_f grey rows and ℓ_u white rows.
- S_N^f and S_N^u consist of sums in $\mathcal{F}_N$ and $\mathcal{U}_N$.

Example: $N = 8$, $d = 3$ with 3 observations trusted ($\ell_f = 3$)

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad S_N = \begin{bmatrix} 8 \\ 3 \\ 5 \\ 3 \\ 8 \\ 4 \\ 4 \\ 9 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 10 \\ 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

Maximum variance (or maximum TVaR)

upper Fréchet bound, comonotonic scenario

- To maximize the variance of S_N: comonotonic scenario on $\mathcal{U}_N$, and the corresponding values of the sums $\tilde{s}_i$ $(i = 1, 2, ..., \ell_u)$
- Average sum $\bar{s} = 5.5$.
- Maximum variance

$$\frac{1}{8}\left(\sum_{i=1}^{3}(s_i - \bar{s})^2 + \sum_{i=1}^{5}(\tilde{s}_i^c - \bar{s})^2\right) \approx 8.75$$

Minimum variance (or minimum TVaR)

Antimonotonicity in d dimensions?
The rearrangement algorithm (RA) of Puccetti & Rüschendorf, 2012 aims to obtain sums with variance as small as possible.

Idea of the RA

- Columns of M are rearranged such that they become anti-monotonic with the sum of all other columns.

$$\forall k \in \{1,2,...,d\}, X_k^a \textit{ antimonotonic with } \sum_{j\neq k} X_j$$

- After each step, $var\left(X_k^a + \sum_{j\neq k} X_j\right) \leqslant var\left(X_k + \sum_{j\neq k} X_j\right)$ where X_k^a is antimonotonic with $\sum_{j\neq k} X_j$

Example: Minimum Variance

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 10 \\ 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

Minimum variance obtained when S_N^u has smallest variance (ideally constant, "mixability")

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$

The minimum variance is
$\frac{1}{8}\left(\sum_{i=1}^{3}(s_i - \bar{s})^2 + \sum_{i=1}^{5}(\tilde{s}_i^m - \bar{s})^2\right) \approx 2.5$

Example $d = 20$

- $(X_1, ..., X_{20})$ independent multivariate normal on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d$$

(for some $\beta \leqslant 50\%$) where q_γ: γ-quantile of N(0,1)

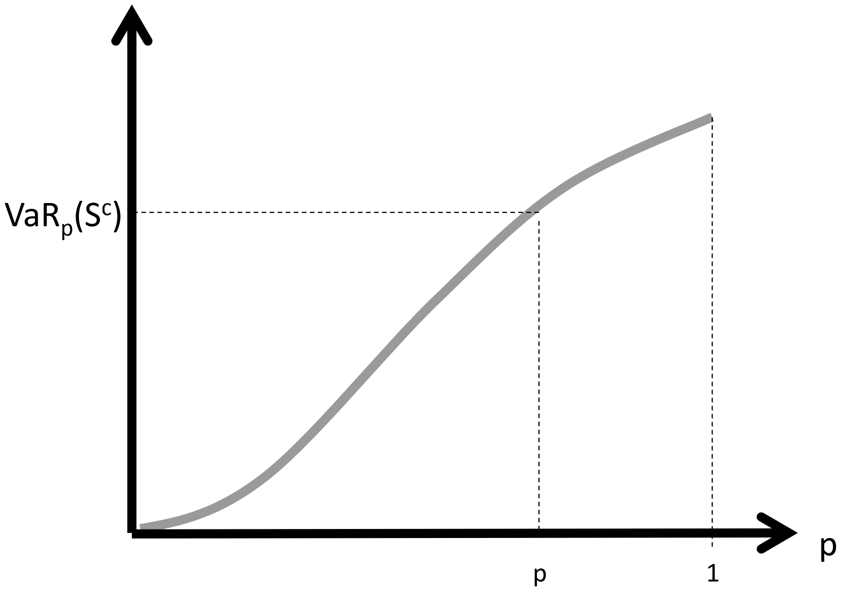
- $\beta = 0\%$: no uncertainty (multivariate assumption)
- $\beta = 50\%$: full uncertainty

$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$p_f \approx 98\%$ $\beta = 0.05\%$	$p_f \approx 82\%$ $\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 50\%$
$\rho = 0$	4.47	(4.4 , 5.65)	(3.89 , 10.6)	(0 , 20)

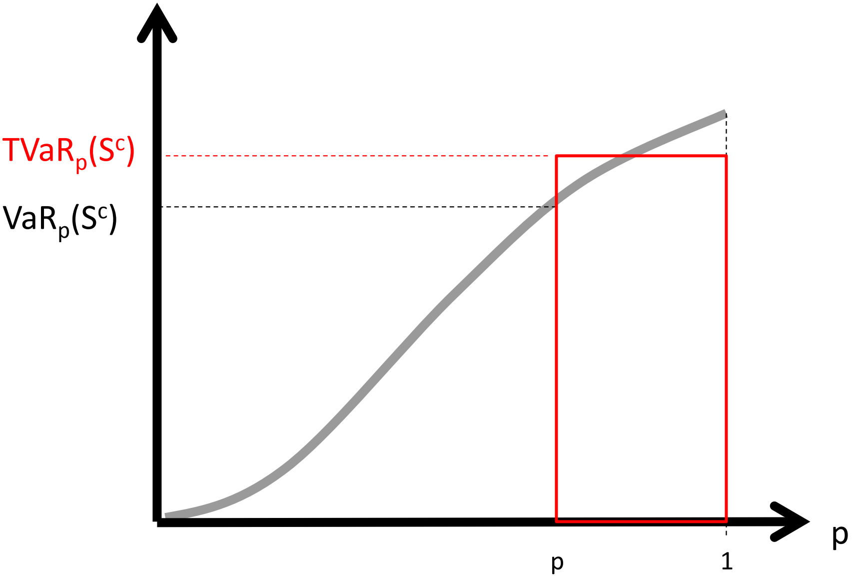
Bounds on Value-at-Risk

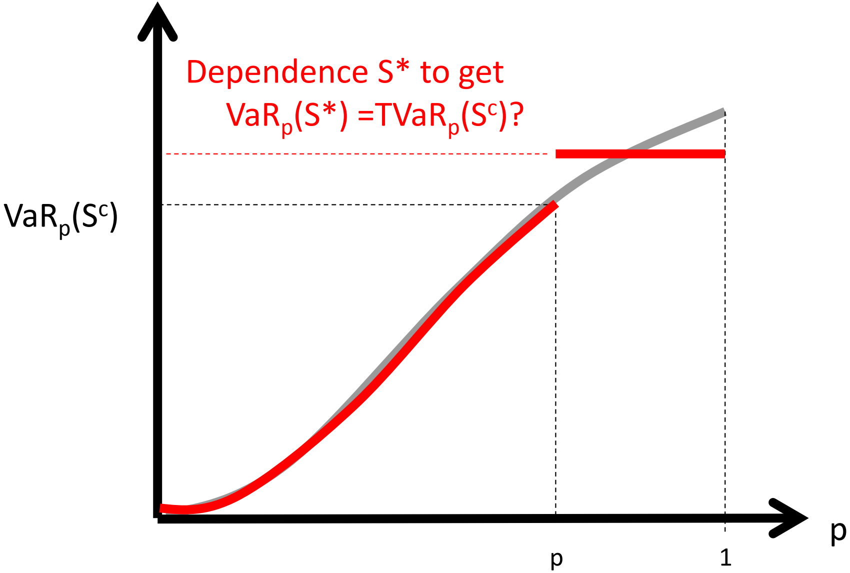
Previous approach works for all risk measures that satisfy convex order... But not for Value-at-Risk

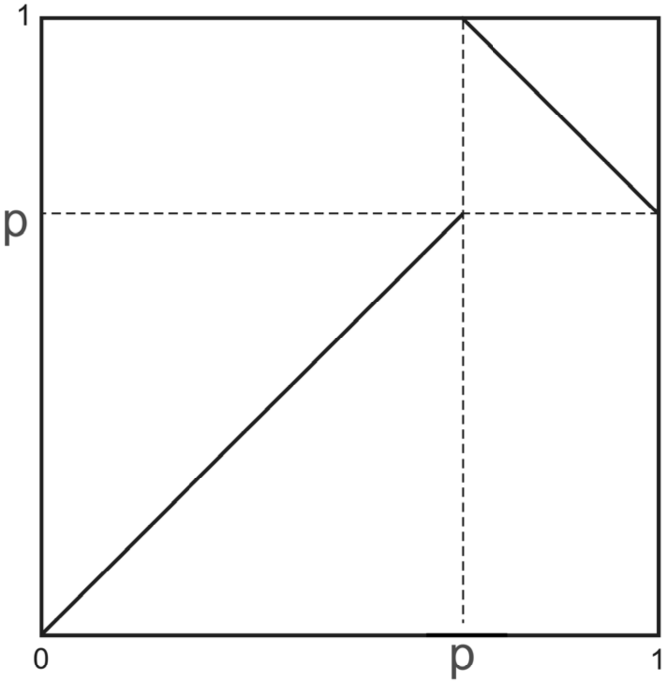
- to maximize VaR_p, the idea is to change the comonotonic dependence of Z_i such that the sum is constant in the tail



VaR_p(S^c)
p
1
p


TVaR_p(S^c)
VaR_p(S^c)
p
1
p


Dependence S* to get
VaRp(S*) =TVaRp(Sc)?
VaRp(Sc)
p
p
1


1
p
0
p
1

Numerical Results, 20 independent $N(0,1)$, $\mathcal{F} = [q_\beta, q_{1-\beta}]^d$

	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$p_f \approx 98\%$ $\beta = 0.05\%$	$p_f \approx 82\%$ $\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
p=95%	12.5	(12.2 , 13.3)	(10.7 , 27.7)	(-2.17 , 41.3)
p=99.95%	25.1	(24.2 , 71.1)	(21.5 , 71.1)	(-0.035 , 71.1)

- $\mathcal{U} = \emptyset$: No uncertainty (multivariate standard normal model).

- **The risk for an underestimation of VaR is increasing in the probability level used to assess the VaR.**
- For VaR at high probability levels ($p = 99.95\%$), **despite all the added information on dependence, the bounds are still wide!**

Conclusions

- Assess model risk with partial information and given marginals (Monte Carlo from fitted model or non-parametrically)
- Algorithm for variance, TVaR and VaR
- Application to the model risk of a portfolio of stocks with market data
- How to choose the trusted area $\mathcal{F}$ optimally?
- N too small: possible to improve the efficiency of the algorithm by re-discretizing using the fitted marginal $\hat{f}_i$.
- Possible to amplify the tails of the marginals by re-discretizing with a probability distortion
- Additional information on dependence can be incorporated
  - variance of the sum (WP with Rüschendorf,Vanduffel)
  - higher moments (WP with Denuit, Vanduffel)

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Second Approach

Model Risk Analytical Bounds

Some Notation

- Define $p_f := P(\mathbb{I} = 1)$ and $p_u := P(\mathbb{I} = 0)$ where

$$\mathbb{I} := \mathbb{1}_{(X_1, X_2, ..., X_d) \in \mathcal{F}} \tag{2}$$

- Let $U \sim \mathcal{U}(0,1)$ independent of the event "$(X_1, X_2, ..., X_d) \in \mathcal{F}$" (so U is independent of $\mathbb{I}$).
- Define $(Z_1, Z_2, ..., Z_d)$ by

$$Z_i = F^{-1}_{X_i | (X_1, X_2, ..., X_d) \in \mathcal{U}}(U), \qquad i = 1, 2, ..., d \tag{3}$$

- All Z_i $(i = 1, 2, ..., d)$ are increasing in U and thus $(Z_1, Z_2, ..., Z_d)$ is comonotonic with known distribution.

Bounds on Variance

Theorem (Bounds on the variance of $\sum_{i=1}^{d} X_i$)

Let $(X_1, X_2, ..., X_d)$ that satisfies properties (i), (ii) and (iii) and let $(Z_1, Z_2, ..., Z_d)$ and $\mathbb{I}$ as defined before.

$$
\begin{aligned}
var\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} E(Z_i)\right) &\leqslant var\left(\sum_{i=1}^{d} X_i\right) \\
&\leqslant var\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} Z_i\right)
\end{aligned}
$$

Bounds on VaR

Theorem (Constrained VaR Bounds for $\sum_{i=1}^{d} X_i$)

Assume $(X_1, X_2, ..., X_d)$ satisfies properties (i), (ii) and (iii), and let $(Z_1, Z_2, ..., Z_d)$, U and $\mathbb{I}$ as defined before. Define the variables L_i and H_i as

$$L_i = LTVaR_U(Z_i) \text{ and } H_i = TVaR_U(Z_i)$$

and let

$$m_p := VaR_p\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} L_i\right)$$
$$M_p := VaR_p\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} H_i\right)$$

Bounds on the Value-at-Risk are $m_p \leqslant VaR_p\left(\sum_{i=1}^{d} X_i\right) \leqslant M_p$.

Value-at-Risk of a Mixture

Lemma

Consider a sum $S = \mathbb{I}X + (1-\mathbb{I})Y$, *where* $\mathbb{I}$ *is a Bernoulli distributed random variable with parameter* p_f *and where the components* X *and* Y *are independent of* $\mathbb{I}$. *Define* $\alpha_* \in [0,1]$ *by*

$$\alpha_* := \inf\left\{\alpha \in (0,1) \mid \exists \beta \in (0,1) \left\{ \begin{array}{l} p_f\alpha + (1-p_f)\beta = p \\ VaR_\alpha(X) \geqslant VaR_\beta(Y) \end{array} \right. \right\}$$

and let $\beta_* = \frac{p - p_f\alpha_*}{1-p_f} \in [0,1]$. *Then, for* $p \in (0,1)$,

$$VaR_p(S) = \max\left\{VaR_{\alpha_*}(X), VaR_{\beta_*}(Y)\right\}$$

Applying this lemma, one can prove a more convenient expression to compute the VaR bounds.

Let us define $T := F^{-1}_{\sum_i X_i|(X_1,X_2,\ldots,X_d)\in\mathcal{F}}(U)$.

Theorem (Alternative formulation of the upper bound for VaR)

Assume $(X_1, X_2, \ldots, X_d)$ satisfies properties (i), (ii) and (iii), and let $(Z_1, Z_2, \ldots, Z_d)$ and $\mathbb{I}$ as defined before.
With $\alpha_1 = \max\left\{0, \frac{p+p_f-1}{p_f}\right\}$ and $\alpha_2 = \min\left\{1, \frac{p}{p_f}\right\}$,
$\alpha_* := \inf\left\{\alpha \in (\alpha_1, \alpha_2) \mid VaR_\alpha(T) \geqslant TVaR_{\frac{p-p_f\alpha}{1-p_f}}\left(\sum_{i=1}^d Z_i\right)\right\}$
When $\frac{p+p_f-1}{p_f} < \alpha_* < \frac{p}{p_f}$,

$$M_p = TVaR_{\frac{p-p_f\alpha_*}{1-p_f}}\left(\sum_{i=1}^d Z_i\right)$$

The lower bound m_p is obtained by replacing "TVaR" by "LTVaR".