Cash Flows and Policyholder Behaviour

in the semi-Markov life insurance setup

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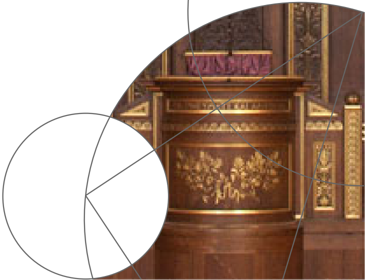
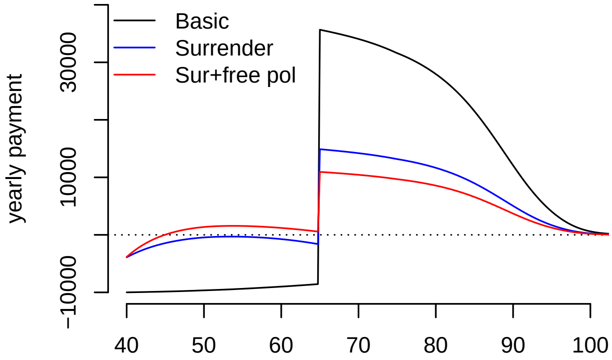


Main Results

1. (Rediscovery of) Kolmogorov's forward integro-diff. eq.
2. Free policy modelling with extra duration eliminated
   by a modified version of Kolmogorov's forward integro-diff. eq.
3. Significant impact from the modelling of policyholder behaviour:

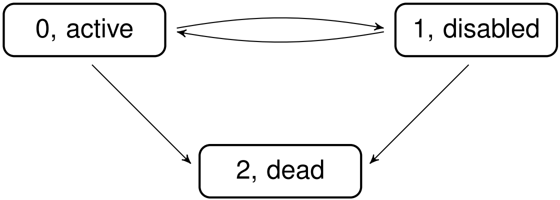


Semi-Markov life insurance setup

- Stochastic process $Z(t) \in \mathcal{J} = \{0, 1, \dots, J\}$.
- Duration in current state $U(t)$.
- Assume $Z(t)$ is semi-Markov $\Leftrightarrow (Z(t), U(t))$ is Markov.
- Transition rates allowed to be duration dependent, $\mu_{ij}(t, u)$.

The *transition probabilities* are duration dependent

$$p_{ij}(t, s, u, v) = P(Z(s) = j, U(s) \leq v \mid Z(t) = i, U(t) = u)$$



Life insurance payments

Time t, duration u:

- While in state j: $b_j(t,u)$,
- Transition from i to j: $b_{ij}(t,u)$.

Conditional on $(Z(t)=i, U(t)=u)$: The (expected) *cash flow* at time s,

$$\mathrm{d}A_{i,u}(t,s) = \sum_{j\in\mathcal{J}} \int p_{ij}(t,s,u,\mathrm{d}v)\left(b_j(s,v) + \sum_{k:k\neq j} \mu_{jk}(s,v) b_{jk}(s,v)\right)\mathrm{d}s.$$

The *prospective reserve*,

$$V_{i,u}(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A_{i,u}(t,s).$$

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- Primary question: How to calculate the distr. $p_{ij}(t,s,u,\mathrm{d}v)$?

Kolmogorov's forward integro-diff. eq.

Theorem For $D(t) = d + t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t,s,u,D(s)) = -\int_0^{D(s)} p_{ij}(t,s,u,\mathrm{d}z)\mu_{j\cdot}(s,z) + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \int p_{ik}(t,s,u,\mathrm{d}z)\mu_{kj}(s,z),$$

$$p_{ij}(t,t,u,d) = 1_{\{i=j\}} 1_{\{u \leq d\}},$$

$$p_{ij}(t,s,u,0) = 0, \quad s > t.$$

Similar results in Hoem (1972) and Helwich (2008).

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One "solve" yields $p_{ij}(t,s,u,D(s))$ for all $s \in (t,T)$ and $j \in \mathcal{J}$.
$\Rightarrow$ solve only once!

The (Danish) setup: With profit products

The Policy

1. policyholder agrees to pay a premium
2. life/pension insurance company guarantees certain benefits

The policy is valued with 2 valuation bases

Technical basis: Safe-side

Determines relation between premium and guarantee.

- Conservative (low) interest rate r^*
- Safe-side mortality rate, disability rate, etc.

Market basis: Best estimate

Determines balance-sheet value of the liabilities (guaranteed payments).

- Market inferred forward interest rate f^r.
- Best estimate mortality rate, disability rate, etc.

Policyholder behaviour: 2 options

Surrender

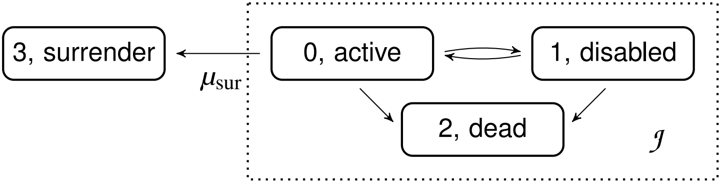
- cancel all future payments, and
- receives policy value, according to the technical basis

Free policy (eqv. paid-up policy)

- cancel all future premiums, and
- the benefits are reduced, according to the technical basis

Payments are calculated on the technical basis:
$\Rightarrow$ Introduces risk on the market basis.
$\Rightarrow$ Market based valuation should include policyholder behaviour.

Surrender modelling

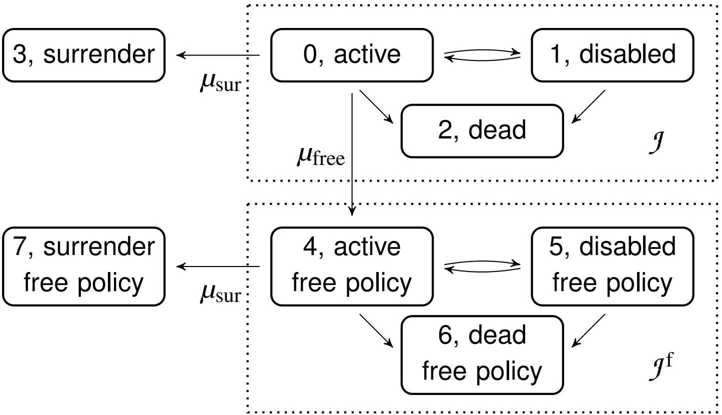


Surrender transition: payment $V^*_{0,0}(t)$

Cash flow

$$\mathrm{d}A^{\mathrm{s}}_{i,u}(t,s)$$
$$= \sum_{j\in\mathcal{J}} \int p_{ij}(t,s,u,\mathrm{d}v)\left(b_j(s,v) + \sum_{k:k\neq j} \mu_{jk}(s,v) b_{jk}(s,v)\right)\mathrm{d}s$$
$$+ \int p_{i0}(t,s,u,\mathrm{d}v)\mu_{\text{sur}}(s,v) V^*_{0,0}(s)\,\mathrm{d}s.$$

State space: Surrender & free policy



Free policy at time t:

- Premiums cancelled
- Future payments reduced by factor $\rho(t)$

New duration $W(t)$: Time since free policy conversion.

Free policy cash flow

Define ρ-modified transition probabilities

$$p_{ij}^{\rho}(t,s,u,z)$$
$$= \int_t^s \int_0^{u+\tau-t} p_{i0}(t,\tau,u,\mathrm{d}v)\mu_{\mathrm{free}}(\tau,v)\rho(\tau)p_{\mathrm{ActFree},j}(\tau,s,0,z)\,\mathrm{d}\tau$$

Free policy cash flow

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Proposition The cash flow is,

$$\mathrm{d}A_{i,u}^{\mathrm{fs}}(t,s) = \mathrm{d}A_{i,u}^{\mathrm{s}}(t,s) + \sum_{j\in\mathcal{J}^{\mathrm{f}}} \int p_{ij}^{\rho}(t,s,u,\mathrm{d}z)\Bigg(b_j^{+}(s,z) + \sum_{\substack{k\in\mathcal{J}^{\mathrm{f}}\\ k\neq j}} \mu_{jk}(s,z) b_{jk}(s,z)^{+}\Bigg)\mathrm{d}s + \int p_{i,\mathrm{ActFree}}^{\rho}(t,s,u,\mathrm{d}z)\mu_{\mathrm{sur}}(s,z)V_{0,0}^{*,+}(s)\,\mathrm{d}s$$

p^ρ forward integro-differential equation

Theorem $p^\rho_{ij}(t,s,u,v)$ satisfy, with $D(s) = d + s - t \geq 0$, $d \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}s} p^\rho_{ij}(t,s,u,D(s)) = 1_{\{j=\mathrm{ActFree}\}} \int p_{i0}(t,s,u,\mathrm{d}z)\mu_{\mathrm{free}}(s,z)\rho(s) - \int_0^{D(s)} p^\rho_{ij}(t,s,u,\mathrm{d}z)\mu_{j.}(s,z) + \sum_{\substack{k\in\mathcal{J}^{\mathrm{f}} \\ k\neq j}} \int p^\rho_{ik}(t,s,u,\mathrm{d}z)\mu_{kj}(s,z)$$

$$p^\rho_{ij}(t,t,u,d) = 0, \quad p^\rho_{ij}(t,s,u,0) = 0, \quad s > t.$$

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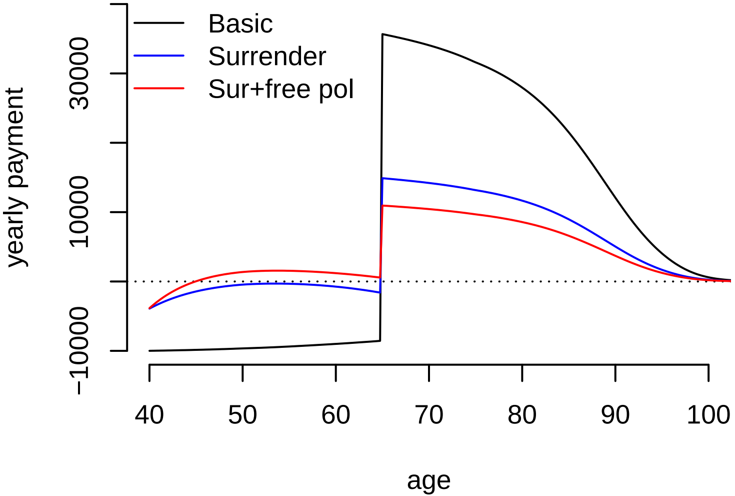
Compare to Kolmogorov forward integro-diff. eq.

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t,s,u,D(s)) = -\int_0^{D(s)} p_{ij}(t,s,u,\mathrm{d}z)\mu_{j.}(s,z) + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \int p_{ik}(t,s,u,\mathrm{d}z)\mu_{kj}(s,z)$$

Total cash flow



Conclusion

- Reviewed Kolmogorov's forward integro-diff. eq.
  for efficient life insurance cash flows, in the semi-Markov setup.
- Cash flows efficiently calculated with policyholder behaviour
  with a modified Kolmogorov forward integro-diff.-eq.
- Policyholder behaviour has a huge effect on cash flows.
  Essential for interest rate sensitivity analysis.
- With policyholder modelling, significantly less interest rate hedging is needed.

References

- K. Buchardt, K. B. Schmidt and T. Møller (2014), *Cash flows and policyholder behaviour in the semi-Markov life insurance setup.* to appear in Scandinavian Actuarial Journal.
- K. Buchardt and T. Møller (2013), *Life insurance cash flows with policyholder behaviour.* Preprint, Department of Mathematical Sciences, University of Copenhagen and PFA Pension.

Thank you!