Asymptotics for ruin probabilities in a discrete-time risk model with dependent financial and insurance risks

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1. Discrete-time insurance risk model.

Let \( \{X_i, i \geq 1\} \) and \( \{Y_i, i \geq 1\} \) be insurance and financial risks and the aggregate net losses

\[
S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j ,
\]

for each positive integer \( n \). The finite and infinite time ruin probabilities are

\[
\psi(x, n) = P \left( \max_{1 \leq k \leq n} S_k > x \right) ,
\]

\[
\psi(x) = \lim_{n \to \infty} \psi(x, n) = P \left( \sup_{n \geq 1} S_n > x \right) ,
\]

where \( x \geq 0 \) is interpreted as the initial capital.
If we denote the product $\theta_i = \prod_{j=1}^{i} Y_j$ in (1), the ruin probabilities $\psi(x, n)$ and $\psi(x)$ represent the tail probabilities of the maximum of randomly weighted sums. In case of independence between $\{X_i, i \geq 1\}$ and $\{\theta_i, i \geq 1\}$, under the presence of heavy-tailed insurance risks, was recently established the asymptotic formula

$$\psi(x, n) \sim \sum_{i=1}^{n} P(X_i \theta_i > x), \; x \to \infty,$$

holds for each fixed $n$, or

$$\psi(x) \sim \sum_{i=1}^{\infty} P(X_i \theta_i > x), \; x \to \infty.$$
1. Dependence structure.

The survival copula $C(u, v)$ is defined by the formula

$$C(u, v) = u + v - 1 + C(1 - u, 1 - v), \ (u, v) \in [0, 1]^2,$$

with respect to the given copula $C(u, v)$. Clearly, the survival copula with respect to $C(u, v)$ can be represented as

$$C(F(x), G(y)) = \mathbf{P}(X > x, Y > y).$$
Assume that the copula function $C(u, v)$ is absolutely continuous. Denote by $C_1(u, v) = \frac{\partial}{\partial u} C(u, v)$, $C_2(u, v) = \frac{\partial}{\partial v} C(u, v)$, $C_{12}(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v)$. Then

$$
\overline{C}_2(u, v) := \frac{\partial}{\partial v} \overline{C}(u, v) = 1 - C_2(1 - u, 1 - v),
$$

and

$$
\overline{C}_{12}(u, v) := \frac{\partial^2}{\partial u \partial v} \overline{C}(u, v) = C_{12}(1 - u, 1 - v).
$$

**Assumption A_1.** (Albrecher et al. 06) There exists a positive constant $M$ such that

$$
\lim_{v \uparrow 1} \lim_{u \uparrow 1} C_{12}(u, v) = \lim_{v \uparrow 1} \lim_{u \uparrow 1} \overline{C}_{12}(1 - u, 1 - v) < M.
$$
Assumption A₂. (Asimit and Badescu 10) The relation

\[ \overline{C}_2(u, v) \sim u \overline{C}_{12}(0+, v), \ u \downarrow 0, \]

holds uniformly on (0, 1).
Clearly, Assumption A₂ is equivalent to

\[ 1 - C_2(u, v) \sim (1 - u) C_{12}(1-, v), \ u \uparrow 1, \]

holds uniformly on (0, 1). Thus, if the copula \( C(u, v) \) of the random vector \((X, Y)\) satisfies Assumptions A₁ and A₂, then the copula of \((X^+, Y)\), denoted by \( C^+(u, v) \), satisfies these two assumptions as well.
Assumption A₃. The relation

\[ C_2(u, v) = 1 - \bar{C}_2(1 - u, 1 - v) \to 0, \quad u \downarrow 0, \]

holds uniformly on \([0, 1]\).

We remark that Assumption A₃ is equivalent to the fact that

\[ \mathbb{P}(X \leq x \mid Y = y) = C_2[F(x), G(y)] \to 0, \quad x \to -\infty, \]

holds uniformly on \(\mathbb{R}\).
We remind the classes of distributions:

1. 
\[ \mathcal{L} = \left\{ F \left| \lim_{x \to \infty} \frac{F(x - y)}{F(x)} = 1, \forall y \in \mathbb{R} \right. \right\} , \]

2. 
\[ \mathcal{D} = \left\{ F \left| \limsup_{x \to \infty} \frac{F(xu)}{F(x)} < \infty, \forall u \in (0, 1) \right. \right\} . \]

3. 
\[ \mathcal{C} = \left\{ F \left| \lim_{u \uparrow 1} \limsup_{x \to \infty} \frac{F(xu)}{F(x)} = 1, \right. \right\} . \]
A distribution $F$ on $\mathbb{R}$ belongs to the class $R_{-\alpha}$, if $\lim \frac{F(xy)}{F(x)} = y^{-\alpha}$ for some $\alpha \geq 0$ and all $y > 0$. It is well known that the following inclusion relationships hold:

$$R_{-\alpha} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{L}$$

Furthermore, for a distribution $F$ on $\mathbb{R}$, denote its upper and lower Matuszewska indices, respectively, by

$$J^+_F = - \lim_{y \to \infty} \frac{\log F^*_y(y)}{\log y} \quad \text{with} \quad F^*_y(y) := \liminf \frac{F(xy)}{F(x)} \quad \text{for} \quad y > 1,$$

$$J^-_F = - \lim_{y \to \infty} \frac{\log F^*_y(y)}{\log y} \quad \text{with} \quad F^*_y(y) := \limsup \frac{F(xy)}{F(x)} \quad \text{for} \quad y > 1.$$
3. Asymptotic results.

**Theorem 1.** In the discrete-time risk model, assume that \( \{(X_i, Y_i), \ i \geq 1\} \) are i.i.d. random vectors with generic random vector \((X, Y)\) satisfying Assumptions A\(_1\)–A\(_3\). If \( F \in \mathcal{C} \) and \( \mathbb{E}Y^p < \infty \) for some \( p > J_F^+ \), then, for each fixed \( n \geq 1 \), it holds that

\[
\psi(x, n) \sim \sum_{i=1}^{n} H_i(x). \tag{7}
\]
Corollary 1.  (1) Under the conditions of Theorem 1, if $F \in R_{-\alpha}$ for some $\alpha \geq 0$, then, for each fixed $n \geq 1$, it holds that

$$
\psi(x, n) \sim \mathbb{E} Y^\alpha_c \frac{1 - (\mathbb{E} Y^\alpha)^n}{1 - \mathbb{E} Y^\alpha} \overline{F}(x),
$$

by convention, $(1 - (\mathbb{E} Y^\alpha)^n)/(1 - \mathbb{E} Y^\alpha) = n$ if $\mathbb{E} Y^\alpha = 1$. 
Theorem 2. Under the conditions of Theorem 1, if $J_F^- > 0$ and $\mathbb{E}Y^p < 1$ for some $p > J_F^+$, then it holds that

$$\psi(x) \sim \sum_{i=1}^{\infty} H_i(x).$$

(9)
Corollary 2. Under the conditions of Theorem 2, if $F \in R_{-\alpha}$ for some $\alpha > 0$, then

$$\psi(x) \sim \frac{EY_\alpha c}{1 - EY_\alpha} F(x).$$

(10)

Moreover, (8) holds uniformly over the integers $\{n \geq 1\}$. 
The last result shows that the asymptotic relation (11) for the finite time probability is uniform over the integers \( \{n \geq 1\} \).

**Theorem 3.** Under the conditions of Theorem 2, the asymptotic relation

\[
\psi(x, n) \sim \sum_{i=1}^{n} H_i(x).
\]  \(11\)

holds uniformly over the integers \( \{n \geq 1\} \).
References


Thank you!