Asymptotics for ruin probabilities in a discrete-time risk model with dependent financial and insurance risks

Yang Yang
School of Mathematics and Statistics, Nanjing Audit University
School of Economics and Management, Southeast University, China

Dimitrios G. Konstantinides
Department of Mathematics, University of the Aegean, Greece

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1. Discrete-time insurance risk model.

Let $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ be insurance and financial risks and the aggregate net losses

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j\,, \tag{1}$$

for each positive integer n. The finite and infinite time ruin probabilities are

$$\psi(x,\, n) = \mathbf{P}\left(\max_{1\leq k\leq n} S_k > x\right)\,, \tag{2}$$

$$\psi(x) = \lim_{n\to\infty} \psi(x,\, n) = \mathbf{P}\left(\sup_{n\geq 1} S_n > x\right)\,, \tag{3}$$

where $x \geq 0$ is interpreted as the initial capital.

If we denote the product $\theta_i = \prod_{j=1}^{i} Y_j$ in (1), the ruin probabilities $\psi(x, n)$ and $\psi(x)$ represent the tail probabilities of the maximum of randomly weighted sums.
In case of independence between $\{X_i,\ i \geq 1\}$ and $\{\theta_i,\ i \geq 1\}$, under the presence of heavy-tailed insurance risks, was recently established the asymptotic formula

$$\psi(x, n) \sim \sum_{i=1}^{n} \mathbf{P}(X_i\,\theta_i > x),\ x \to \infty\,, \tag{4}$$

holds for each fixed n, or

$$\psi(x) \sim \sum_{i=1}^{\infty} \mathbf{P}(X_i\,\theta_i > x)\,,\ x \to \infty\,. \tag{5}$$

1. Dependence structure.

The survival copula $\overline{C}(u,\, v)$ is defined by the formula

$$\overline{C}(u,\, v) = u + v - 1 + C(1-u,\, 1-v)\,,\ (u,\, v) \in [0,\, 1]^2\,,$$

with respect to the given copula $C(u,v)$. Clearly, the survival copula with respect to $C(u,v)$ can be represented as

$$\overline{C}(\overline{F}(x), \overline{G}(y)) = \mathbf{P}(X > x, Y > y)\,.$$

Assume that the copula function $C(u, v)$ is absolutely continuous. Denote by $C_1(u, v) = \frac{\partial}{\partial u} C(u, v)$, $C_2(u, v) = \frac{\partial}{\partial v} C(u, v)$, $C_{12}(u, v) = \frac{\partial^2}{\partial u\, \partial v} C(u, v)$. Then

$$\overline{C}_2(u, v) := \frac{\partial}{\partial v} \overline{C}(u, v) = 1 - C_2(1-u, 1-v)\,,$$

and

$$\overline{C}_{12}(u, v) := \frac{\partial^2}{\partial u\, \partial v} \overline{C}(u, v) = C_{12}(1-u, 1-v)\,.$$

Assumption A$_1$. (Albrecher et al. 06) *There exists a positive constant M such that*

$$\limsup_{v \uparrow 1} \limsup_{u \uparrow 1} C_{12}(u, v) = \limsup_{v \uparrow 1} \limsup_{u \uparrow 1} \overline{C}_{12}(1-u, 1-v) < M\,.$$

Assumption A_2. (Asimit and Badescu 10) *The relation*

$$\overline{C}_2(u,\, v) \sim u\,\overline{C}_{12}(0+,\, v)\,,\; u \downarrow 0\,,$$

holds uniformly on $(0,\, 1)$.

Clearly, Assumption A_2 is equivalent to

$$1 - C_2(u,\, v) \sim (1-u)\, C_{12}(1-, v)\,,\; u \uparrow 1\,,$$

holds uniformly on $(0,\, 1)$. Thus, if the copula $C(u,\, v)$ of the random vector $(X,\, Y)$ satisfies Assumptions A_1 and A_2, then the copula of $(X^+,\, Y)$, denoted by $C^+(u,\, v)$, satisfies these two assumptions as well.

Assumption A$_3$. *The relation*

$$C_2(u,\, v) = 1 - \overline{C}_2(1-u,\, 1-v) \to 0\,,\; u \downarrow 0\,,$$

holds uniformly on $[0,\, 1]$.

We remark that Assumption A$_3$ is equivalent to the fact that

$$\mathbf{P}(X \le x \mid Y = y) = C_2[F(x),\, G(y)] \to 0\,,\; x \to -\infty\,, \tag{6}$$

holds uniformly on $\mathbb{R}$.

We remind the classes of distributions:

1.

$$\mathcal{L} = \left\{ F \,\middle|\, \lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1,\, \forall\, y \in \mathbb{R} \right\},$$

2.

$$\mathcal{D} = \left\{ F \,\middle|\, \limsup_{x\to\infty} \frac{\overline{F}(xu)}{\overline{F}(x)} < \infty,\, \forall\, u \in (0,\, 1) \right\}.$$

3.

$$\mathcal{C} = \left\{ F \,\middle|\, \lim_{u\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xu)}{\overline{F}(x)} = 1, \right\}.$$

A distribution F on $\mathbb{R}$ belongs to the class $R_{-\alpha}$, if $\lim \overline{F}(x\,y)/\overline{F}(x) = y^{-\alpha}$ for some $\alpha \geq 0$ and all $y > 0$. It is well known that the following inclusion relationships hold:

$$R_{-\alpha} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{L}$$

Furthermore, for a distribution F on $\mathbb{R}$, denote its upper and lower Matuszewska indices, respectively, by

$$J_F^{+} = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y} \ \text{ with } \ \overline{F}_*(y) := \liminf \frac{\overline{F}(x\,y)}{\overline{F}(x)} \ \text{ for } \ y > 1\,,$$

$$J_F^{-} = -\lim_{y\to\infty} \frac{\log \overline{F}^*(y)}{\log y} \ \text{ with } \ \overline{F}^*(y) := \limsup \frac{\overline{F}(x\,y)}{\overline{F}(x)} \ \text{ for } \ y > 1\,.$$

Clearly, $F \in \mathcal{D}$ is equivalent to $J_F^{+} < \infty$. For each $i \geq 1$, denote by H_i the distribution of $X_i \prod_{j=1}^{i} Y_j,\ j \geq 1$ and let $H_1 = H$.

3. Asymptotic results.

Theorem 1. *In the discrete-time risk model, assume that* $\{(X_i, Y_i),\ i \geq 1\}$ *are i.i.d. random vectors with generic random vector* (X, Y) *satisfying Assumptions* A_1*–*A_3*. If* $F \in \mathcal{C}$ *and* $\mathbf{E}Y^p < \infty$ *for some* $p > J_F^+$*, then, for each fixed* $n \geq 1$*, it holds that*

$$\psi(x, n) \sim \sum_{i=1}^{n} \overline{H}_i(x)\,. \tag{7}$$

Corollary 1. *(1) Under the conditions of Theorem 1, if* $F \in R_{-\alpha}$ *for some* $\alpha \geq 0$*, then, for each fixed* $n \geq 1$*, it holds that*

$$\psi(x,\, n) \sim \mathbf{E}Y_c^{\alpha} \frac{1 - (\mathbf{E}Y^{\alpha})^n}{1 - \mathbf{E}Y^{\alpha}} \overline{F}(x)\,, \tag{8}$$

by convention, $(1 - (\mathbf{E}Y^{\alpha})^n)/(1 - \mathbf{E}Y^{\alpha}) = n$ *if* $\mathbf{E}Y^{\alpha} = 1$*.*

Theorem 2. *Under the conditions of Theorem 1, if $J_F^- > 0$ and $\mathbf{E}Y^p < 1$ for some $p > J_F^+$, then it holds that*

$$\psi(x) \sim \sum_{i=1}^{\infty} \overline{H}_i(x) \,. \tag{9}$$

Corollary 2. *Under the conditions of Theorem 2, if $F \in R_{-\alpha}$ for some $\alpha > 0$, then*

$$\psi(x) \sim \frac{\mathbf{E}Y_c^\alpha}{1 - \mathbf{E}Y^\alpha}\, \overline{F}(x)\,. \tag{10}$$

Moreover, (8) holds uniformly over the integers $\{n \geq 1\}$.

The last result shows that the asymptotic relation (11) for the finite time probability is uniform over the integers $\{n \geq 1\}$.

Theorem 3. *Under the conditions of Theorem 2, the asymptotic relation*

$$\psi(x, n) \sim \sum_{i=1}^{n} \overline{H}_i(x) \,. \tag{11}$$

holds uniformly over the integers $\{n \geq 1\}$.

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Thank you!