General price bounds for discrete and continuous arithmetic Asian options

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Outline

- Asian Options
  - Pricing Problem
  - Applications
- Objectives
- Literature Review
- General Lower Bound for Asian Option: Pricing Performance

Pricing problem I

- Payoff depends on the average price of some underlying asset S monitored over a pre-specified period of time $[0, T]$
- Prevalent case of arithmetic average:
  - Discrete average $A_T := \frac{1}{N+1} \sum_{j=0}^{N} S_{\Delta k}$
  - Continuous average $A_T := \frac{1}{T} \int_0^T S_t dt$
- Time-0 price of option

$$V_0 := e^{-rT} \mathbb{E}\left[\left(A_T - K' S_T - K\right) \mathbf{1}_B\right]$$

  - Event set $B := \{A_T > K' S_T + K\}$
  - Fixed strike option: $K > 0$ and $K' = 0$
  - Floating strike option: $K = 0$ and $K' > 0$
  - Price of put-type option: standard put-call parity

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Pricing problem II

The problem

- Pricing of arithmetic Asian options (in general, evaluation of expectations of non-linear functions of the arithmetic average) does not admit true analytical solutions, even under the lognormal model, as the distribution law of the arithmetic average is not known analytically

Some applications of Asian options I

- Raised popularity of arithmetic Asian options among derivatives traders and risk managers:
  - Smooth possible market manipulations
  - Volatility reduction
- Boosted trading activity in highly volatile markets:
  - NYMEX and ICE: e.g., Brent and WTI crude oil average options
  - LME: traded average options on metals (e.g., copper grade A, high grade primary aluminium, etc.)
  - Freight options in dry bulk market (see Nomikos et al., 2013)
- Weighted arithmetic averages appear in technical analysis and algorithmic trading (see Zhu and Zhou, 2009)
- Volume-weighted average price (VWAP): weights given by % trading volumes pertinent to the different transactions (see Novikov and Kordzakhia, 2013)

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Main contributions I

- Main objective of this research is to develop a pricing formula, which is
  - simple
  - accurate
  - fast
- Proposed method can be applied to:
  - Pure jump Lévy models, Merton's normal and Cai and Kou's generalized hyperexponential jump diffusions
  - Affine models with/out jumps in the asset price/stochastic volatility dynamics (e.g., see Heston, 1993, Bates, 1996, Duffie et al., 2000, Carr et al., 2003, Carr and Wu, 2004, Barndorff-Nielsen–Shephard model, Stein–Stein and Schöbel–Zhu model)
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Additional features:

- Application to fixed strike and floating strike options (symmetry relations not available beyond exponential Lévy models)
- Application to discrete and continuous averages
- Extension to option price sensitivities (Greeks)
- Extension to Australian option: payoff depends on A_T/S_T (see Ewald et al., 2013)
- Problem dimension unaffected by additional random volatility factor
- Theoretical upper bound to error from the price approximation using the lower bound is available and can be calculated numerically

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Literature review I

- Large volume of literature devoted to pricing of Asian options.
- P(I)DE methods:
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- Recent advances on pricing Asian options under the CEV diffusion: Cai, Li, and Shi (2013b), Sesana et al. (2014), Cai, Kou, and Song (2013a)

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- Recall

$$B = \{A_{\Delta N} > K' S_{\Delta N} + K\}$$

- Focus on discrete arithmetic Asian call with fixed strike ($K > 0$ and $K' = 0$)
- Apply principles set out in Curran (1994) and Rogers and Shi (1995):

$$V_0 = e^{-rT}\mathbb{E}\left[(A_{\Delta N} - K)\mathbf{1}_B\right] \geq \mathrm{LB}_0 := e^{-rT}\mathbb{E}\left[(A_{\Delta N} - K)\mathbf{1}_{\bar{B}}\right]$$

for some event set $\bar{B} \subset \Omega$

- $A_{\Delta N} \leq K$ in $\bar{B} \setminus B$
- Choose event set $\bar{B}$ that
  - is close proxy to the true event set B (ensures tightness of the bound)
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Pricing performance under different model assumptions I

Table : LB: optimized lower bound. APRE: average % relative error of LB w.r.t. CVMC simulation price estimate (with LB itself used as control variate), obtained across strikes $K \in \{90, 100, 110\}$ and no. of monitoring dates $n \in \{12, 50, 250\}$. Model features: J: jumps in asset price, SV: stochastic volatility, SVJ: jumps in stoch. vol. process.

Model	J	SV	J&SV	J&SVJ	APRE
Lognormal	✗	✗	✗	✗	0.014%
CEV	✗	✗	✗	✗	0.022%
Gen. exp. Lévy	✓				0.034%
Heston		✓			0.021%
Bates			✓		0.021%
Duffie et al.				✓	0.027%
BNS-Γ				✓	0.032%
BNS-IG				✓	0.021%

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