# A General Approach for Drawdown (Drawup) Risks of Time-Homogeneous Markov Processes[1]

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[1]Based on a joint work with David Landriault (University of Waterloo) and Hongzhong Zhang (Columbia University)

# Outline

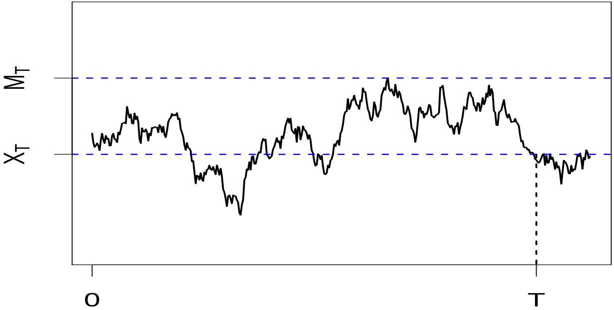
- Introduction of drawdown and its applications
- Main results

# Definition

- Drawdown measures the decline in value from the historical peak for an investment, fund or commodity.
- Consider a stochastic process $X = \{X_t : t \geq 0\}$, the magnitude of drawdown at time $T$ is defined by

$$Y_T := M_T - X_T,$$

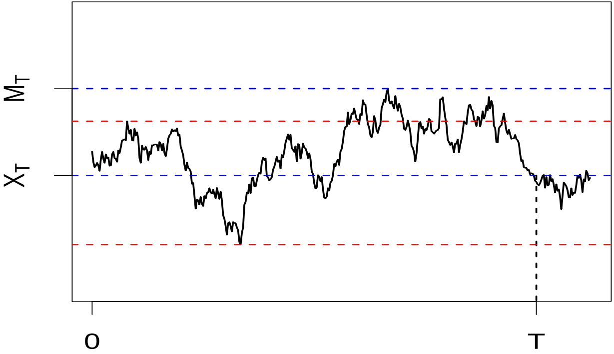
where $M_T = \sup_{0 \leq t \leq T} X_t$



# Maximum drawdown

- The maximum drawdown up to time $T$ is defined by

$$\sup_{0 \le t \le T} Y_t = \sup_{0 \le t \le T} \{M_t - X_t\}$$



# Maximum drawdown (cont.)

- The first time the magnitude of drawdown exceeds a pre-specified level $a > 0$ is denoted by

$$\tau_a := \inf\{t > 0 : Y_t \geq a\}.$$

- (Maximum drawdown before time $T$ exceeds $a$) $\Longleftrightarrow$ ($\tau_a \leq T$)

# Who cares about drawdown?

- mutual funds managers
- financial mathematicians
- statisticians
- probabilists
- ...
- myself :)

# Applications: mutual funds

*"The mutual fund industry and many investment professionals have a well-guarded secret they do not want the investing public to know about: drawdowns. Drawdowns are, in our opinion, the single most important determinant of investing success or failure for most investors.*

*One of the worst characteristics of drawdowns is that they frequently strike like tornados. They hit quickly, without warning, and cause immense damage. It's often difficult to realize their devastation until after they have struck."*

Greg Miller, CPA, CEO
Wellesley Investment Advisors, Inc.
Wellesley, Massachusetts
April 2006 Issue of Investment Advisor

# Applications: mutual funds (cont.)

- Frequently quoted performance measures (Schuhmacher and Eling, 2011)
  - Calmar ratio $= \frac{\text{annual rate of return}}{\text{maximum drawdown}}$
  - Sterling ratio $= \frac{\text{annual rate of return}}{\text{average of maximum drawdowns}}$
  - Burke ratio, Martin ratio, Pain ratio, etc.
- Drawdown is an alternative measurement for volatility.

# Applications: financial mathematics

- Portfolio selection and optimization: Grossman and Zhou (1993), Pospisil and Vecer (2010), Cherny and Obloj (2013), Sekine (2013)
- Option pricing: Russian option

$$
\begin{aligned}
\text{Payoff} &= \sup_{\tau} e^{-a\tau}\{K, M_\tau\} \\
\tau^* &= \inf\{t \geq 0 : Y_t \geq k^*\} \text{ for some } k^*
\end{aligned}
$$

- Insurance: Carr et al. (2011), De Finetti dividend problem

# Applications: statistics

- Change point detection: Hadjiliadis and Moustakides (2006), Khan (2008)
- Queueing: Asmussen (1989), Borovkov (1976), Prabhu (1997)

# Applications: probability

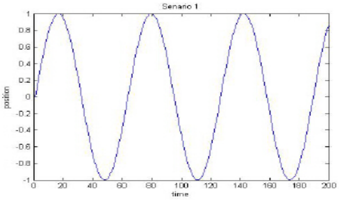
- Drawdown and drawup processes are usually referred as reflected processes

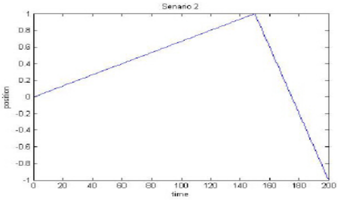
$$Y_t = M_t - X_t \text{ and } \hat{Y}_t = X_t - m_t$$

- Time-homogeneous diffusion processes: Taylor (1975), Lehoczky (1977), Magdon et al. (2004), Pospisil et al. (2009)

- Lévy processes: Asmussen et al. (2004), Pistorius (2004), Mijatovic and Pistorius (2012), Ivanovs and Palmowski (2012)

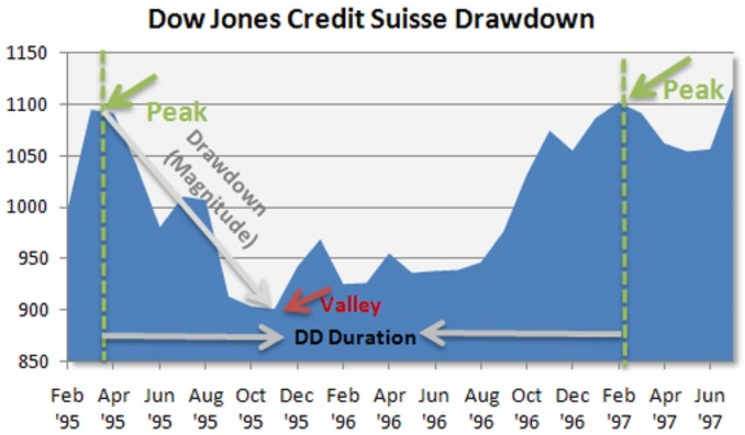
- In addition to the magnitude, there are other two aspects of drawdowns: frequency and duration.




- Frequency of drawdowns: Landriault, **L.**, Zhang (2015)

# Duration of drawdowns




- Duration of drawdowns: Landriault, **L.**, Zhang (2014)

# Outline

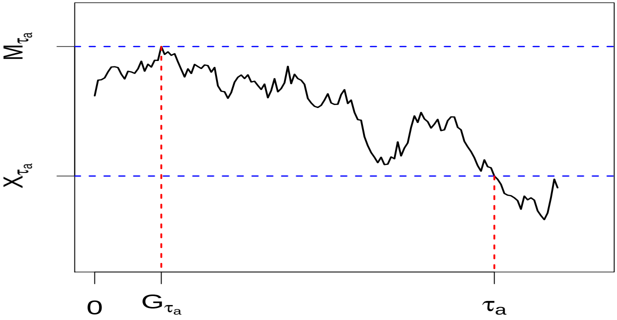
- Introduction of drawdown and its applications
- Main results

- Interested in the joint law of $(\tau_a, M_{\tau_a}, Y_{\tau_a}, G_{\tau_a})$, where

$$G_{\tau_a} = \sup\{0 \le t < \tau_a : M_t = X_t\}$$

is the last time at maximum

- $G_{\tau_a}$ is the turning point from rising to crashing



# Methodologies

- Previous approaches (spectrally negative Lévy models and time-homogeneous diffusion models)
  - Ito excursion theory: Avram et al. (2004), Pistorius (2004), Mijatovic and Pistorius (2012)
  - Martingale theory: Taylor (1975), Asmussen et al. (2004), Nguyen-Ngoc and Yor (2005)
  - Approximation approach: Lehoczky (1977), **L.** et al. (2013), Zhang (2015)

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- Our approach (bound approach)
  - Two steps: constructing tight bounds + taking limits (some regularity assumptions)
  - Advantages: simple, strict, general
  - Building direct connections between drawdown problems and exit problems

# Notation of exit times

- Consider a time-homogeneous Markov process $X$
- Define the first passage times

$$T_x^+ = \inf\{t \geq 0 : X_t > x\} \text{ and } T_x^- = \inf\{t \geq 0 : X_t < x\}$$

- For $u \leq x \leq v$, define two functions on two-sided exits

$$B^{(q,s)}(x;u,v) = \mathbb{E}_x[e^{-qT_v^+ - s(X_{T_v^+} - v)} 1_{\{T_v^+ < T_u^-\}}]$$
$$C^{(q,s)}(x;u,v) = \mathbb{E}_x[e^{-qT_u^- - s(u - X_{T_u^-})} 1_{\{T_u^- < T_v^+\}}]$$

- Analytically tractable models for $B$ and $C$: one-sided Lévy, Kou's double-exponential jump diffusion, meromorphic Lévy, time-homogeneous diffusion, etc.

# Key lemma

## Lemma

*For any* $\varepsilon \in (0, a)$, *we have*

$$\mathbb{E}_x[e^{-qT^+_{x+\varepsilon}} 1_{\{T^+_{x+\varepsilon} < \tau_a\}}] \leq B^{(q,0)}(x; x-a, x+\varepsilon)$$

$$\mathbb{E}_x[e^{-qT^+_{x+\varepsilon}} 1_{\{T^+_{x+\varepsilon} < \tau_a\}}] \geq B^{(q,0)}(x; x+\varepsilon-a, x+\varepsilon)$$

*and for* $\delta \geq 0$,

$$\mathbb{E}_x[e^{-q\tau_a - s(Y_{\tau_a}-a) - \delta(M_{\tau_a}-x)} 1_{\{\tau_a < T^+_{x+\varepsilon}\}}] \leq e^{s\varepsilon} C^{(q,s)}(x; x+\varepsilon-a, x+\varepsilon)$$

$$\mathbb{E}_x[e^{-q\tau_a - s(Y_{\tau_a}-a) - \delta(M_{\tau_a}-x)} 1_{\{\tau_a < T^+_{x+\varepsilon}\}}] \geq e^{-(s+\delta)\varepsilon} C^{(q,s)}(x; x-a, x+\varepsilon$$

# Main result 1

**Assumption (1)**

*The following limits exist and are equal*

$$K_a^{(q,s)} := \lim_{\varepsilon\downarrow 0} \frac{C^{(q,s)}(0;\varepsilon - a,\varepsilon)}{1 - B^{(q,0)}(0;\varepsilon - a,\varepsilon)} = \lim_{\varepsilon\downarrow 0} \frac{C^{(q,s)}(0;-a,\varepsilon)}{1 - B^{(q,0)}(0;-a,\varepsilon)}.$$

**Theorem**

*Consider a general Lévy process $X$ satisfying Assumption (1). Then*

$$\mathbb{E}[e^{-q\tau_a - s(Y_{\tau_a} - a)}] = K_a^{(q,s)}.$$

- In particular, if $X$ is spectrally negative Lévy, we recover Theorem 1 of Avram et al. (2004) and Proposition 2 of Pistorius (2004).

# Main result 2

## Assumption (2)

*The following limits exist and satisfy*

$$\begin{aligned} b_a^{(q,0)}(x) &:= \lim_{\varepsilon \downarrow 0} \frac{1 - B^{(q,0)}(x; x-a, x+\varepsilon)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{1 - B^{(q,0)}(x; x+\varepsilon-a, x+\varepsilon)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{1 - B^{(q,0)}(x-\varepsilon; x-a, x)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{1 - B^{(q,0)}(x-\varepsilon; x-\varepsilon-a, x)}{\varepsilon} \end{aligned}$$

$$\begin{aligned} c_a^{(q,s)}(x) &:= \lim_{\varepsilon \downarrow 0} \frac{C^{(q,s)}(x; x-a, x+\varepsilon)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{C^{(q,s)}(x; x+\varepsilon-a, x+\varepsilon)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{C^{(q,s)}(x-\varepsilon; x-a, x)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{C^{(q,s)}(x-\varepsilon; x-\varepsilon-a, x)}{\varepsilon} \end{aligned}$$

# Main result 2 (cont.)

**Theorem**

*Consider a spectrally negative time-homogeneous Markov process $X$ satisfying Assumption (2). Then,*

$$\mathbb{E}_x\left[e^{-q\tau_a - s(Y_{\tau_a}-a) - \delta(M_{\tau_a}-x)}\right] = \int_x^\infty e^{-\int_x^y \left(\delta + b_a^{(q,0)}(z)\right)\mathrm{d}z} c_a^{(q,s)}(y)\mathrm{d}y$$

- In particular, if $X$ is a time-homogeneous diffusion, we recover results of Lehoczky (1977) by allowing dowside jumps and incorporating the law of $Y_{\tau_a}$.

# Main result 3

Consider a Lévy process with two-sided jumps

$$X_t = \tilde{X}_t + \sum_{i=1}^{N_t^+} J_i^+$$

- $\tilde{X}$ is a spectrally negative Lévy process with Gaussian coefficient $\sigma > 0$
- The Lévy measure $\Pi$ satisfying $\int_{(-1,0)} |x|\Pi(\mathrm{d}x) < \infty$
- $N^+$ is a Poisson process with arrival rate

# Main result 3 (cont.)

**Theorem**

*Consider a Lévy process $X$ of the above form. Then,*

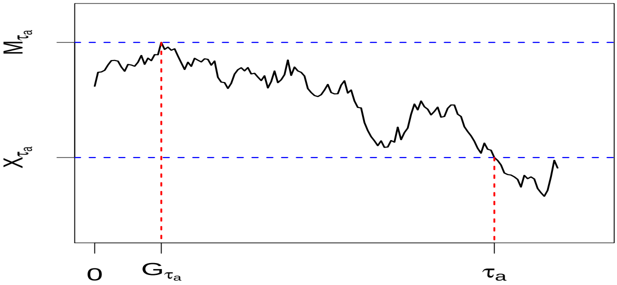
$$\mathbb{E}\left[e^{-q\tau_a - rG_{\tau_a} - s(Y_{\tau_a}-a) - \delta M_{\tau_a}}\right] = \frac{\lim_{\varepsilon\downarrow 0}\frac{C^{(q,s)}(a-\varepsilon;0,a)}{\varepsilon}}{\delta + \lim_{\varepsilon\downarrow 0}\frac{1-B^{(q+r,\delta)}(a-\varepsilon;0,a)}{\varepsilon}}$$

**Corollary**

$(G_{\tau_a}, M_{\tau_a})$ *is independent of* $(\tau_a - G_{\tau_a}, Y_{\tau_a})$

# Independence of drawdown estimates

- We showed $(G_{\tau_a}, M_{\tau_a})$ is independent of $(\tau_a - G_{\tau_a}, Y_{\tau_a})$ for this particular Lévy model with two-sided jumps
- Landriault, **L.**, Zhang (2014) shows that $(G_{\tau_a}, M_{\tau_a})$ is independent of $(\tau_a - G_{\tau_a}, Y_{\tau_a}, Y_{\tau_a-})$ for spectrally negative Lévy processes
- Rising part and crashing part of drawdowns are independent in both time and level scale!!!



**Conjecture**

*$(G_{\tau_a}, M_{\tau_a})$ is independent of $(\tau_a - G_{\tau_a}, Y_{\tau_a}, Y_{\tau_a-})$ for general Lévy processes*

Thank you for your attention!