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Discrete-Time Semi-Markov Random Evolutions: Theory and Applications in Finance

Nikolaos Limnios & Anatoliy Swishchuk

Université de Technologie de Compiègne, France
University of Calgary, Canada

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This talk presents a new model for a stock price in the form of a geometric Markov renewal process (GMRP) which is one of many examples of discrete-time semi-Markov random evolutions (DTSMRE). We study asymptotic properties of the DTSMREs, namely, averaging, diffusion approximation and normal deviations by martingale weak convergence method. As applications we present European call option pricing formula for GMRP.

Random Evolutions

- *Random Evolutions (RE)* are operator dynamical systems (in Banach or Hilbert spaces) where an operator depends on some stochastic process, Markov, semi-Markov, Lévy processes, etc. (see Korolyuk & Swishchuk (1992, 1995), Korolyuk & Limnios (2005))
- For example:

$$\frac{d}{dt}\Phi(t) = C(z(t))\Phi(t)$$

Random Evolutions: Applications

REs have many applications:

- traffic theory
- storage theory
- risk theory
- biomathematics
- **financial mathematics**
- many others

In this talk we shall concentrate on financial applications of REs: *geometric Markov renewal process* (GMRP) as a model for stock price.

Some Literature: DTSMRE

1. **Markov chain case**
    - Keepler (1998);
    - Skorokhod, Hoppensteadt, Salehi (2002);
    - Yin & Zhang (2005)
2. **Embedded Markov case**
    - Koroliuk & Swishchuk (1992, 1995);
    - Koroliuk & Limnios (2005);
    - Swishchuk & Wu (2003)
3. **Semi-Markov chain case**
    - Limnios (2010, 2011)
    - Limnios & Swishchuk (2013)

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Markov Renewal Chains / Semi-Markov Chains

Markov renewal process $(x_n, \tau_n)_{n=0,1,2,\ldots}$

$$\begin{aligned}\mathbb{P}(x_{n+1} \in B, \tau_{n+1} - \tau_n = k \mid x_0, \ldots, x_n; \tau_1, \ldots, \tau_n) \\ = \mathbb{P}(x_{n+1} \in B, \tau_{n+1} - S_n = k \mid x_n)\end{aligned}$$

Semi-Markov Chain $z_k = x_{\tau_{\nu_k}}$, $k \in \mathbb{N}$, where $\nu_k = \{n : \tau_n \leq k\}$.
Semi-Markov kernel

$$\begin{aligned} q(x, B, k) &= \mathbb{P}(x_{n+1} \in B, \tau_{n+1} - \tau_n = k | x_n = x) \\ &= P(x, B) f_{xy}(k) \end{aligned}$$

Transition kernel of the EMC (x_n): $P(x, B)$
Conditional distribution of sojourn time: $f_{xy}(k)$, and $h_x(k) = q(x, E, k)$.

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The semi-Markov setting

Let us consider an ergodic SMC z with state space $(E, \mathcal{E})$; semi-Markov kernel q and ergodic probability π.
Define the backward recurrence chain γ, by

$$\gamma_k := k - \tau_{\nu_k}.$$

For the MC (z, γ) define:

- $P^\sharp$ the transition operator,
- $\pi^\sharp$ the stationary probability, and

Let Π be the stationary projection operator, i.e.,

$$\Pi\varphi(x,s) = \sum_{\ell\geq 0}\int_E \pi^\sharp(dy\times\{\ell\})\varphi(y,\ell)\mathbf{1}(x,s).$$

The potential operator R_0 of $Q^\sharp := P^\sharp - I$, i.e.,

$$R_0 = (Q^\sharp + \Pi)^{-1} - \Pi = \sum_{n\geq 0}[(P^\sharp)^n - \Pi].$$

And $\mathcal{F}_k := \sigma\{z_\ell, \gamma_\ell; \ell \leq k\}$, $k \in \mathbb{N}$.

A Discrete-Time Semi-Markov Random Evolution

Let $\mathbf{B}$ be a Banach space and a family of contraction operators $D(x), x \in E$, and I the identity operator.
Define now the DTSMRE $\Phi_k, k \in \mathbb{N}$, on $\mathbf{B}$ by

$$\Phi_k \varphi = D(z_k)D(z_{k-1}) \cdots D(z_2)D(z_1)\varphi, \quad k \geq 1, \quad \textit{and} \quad \Phi_0 = I.$$

For $\varphi \in \mathbf{B}_0 = \cap_{x \in E} \mathcal{D}(D(x))$.
Thus we have

$$\Phi_k = D(z_k)\Phi_{k-1}.$$

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Stochastic Approximation

We consider series schemes: Φ^{ε}, $\varepsilon > 0$,
for perturbing operators: $D^{\varepsilon}(x)$

- Average approximation
- Diffusion approximation
- Diffusion approximation with equilibrium

Averaging

Assumptions:

A1: The MC $(z_k, \gamma_k, k \in \mathbb{N})$ is uniformly ergodic with ergodic distribution $\pi^\sharp(B \times \{k\}), B \in \mathcal{E}, k \in \mathbb{N}$.

A2: The moments $m_2(x), x \in E$, are uniformly bounded, and $\sup_x \sum_{k \geq T} k^2 h_x(k) \to 0$, as $T \to \infty$.

A3: The perturbed operators $D^\varepsilon(x)$ have the following representation on B

$$D^\varepsilon(x) = I + \varepsilon D_1(x) + \varepsilon D_0^\varepsilon(x),$$

where operators $D_1(x)$ on B are closed and $B_0 := \cap_{x \in E} \mathcal{D}(D_1(x))$ is dense in B, $\overline{B_0} = B$. Operators $D_0^\varepsilon(x)$ are negligible, i.e., $\lim_{\varepsilon \to 0} \|D_0^\varepsilon(x)\varphi\| = 0$ for any $\varphi \in \mathbb{B}_0$.

A4: We have: $\int_E \pi(dx) \|D_1(x)\varphi\|^2 < \infty$.

Averaging

Assumptions (cont'd):

A5: There exists Hilbert spaces H and H^* such that compactly embedded in Banach spaces B and B^*, respectively, where B^* is a dual space to B.

A6: Operators $D^\varepsilon(x)$ and $(D^\varepsilon)^*(x)$ are contractive on Hilbert spaces H and H^*, respectively.

Averaging

THEOREM

Under Assumptions A1-A6, the following weak convergence takes place

$$\Phi^{\varepsilon}_{[t/\varepsilon]} \Longrightarrow \overline{\Phi}(t), \quad \varepsilon \to 0,$$

where the limit process $\overline{\Phi}(t)$ *is determined by the following equation*

$$\overline{\Phi}(t)\varphi - \varphi - \int_0^t \mathbb{L}\overline{\Phi}(s)\varphi ds = 0, \quad 0 \le t \le T,$$

and the limiting operator is given by:

$$\mathbb{L}\Pi = \Pi D_1(\cdot)\Pi.$$

Diffusion approximation

Assumptions:

D1: Let us assume that the perturbed operators $D^{\varepsilon}(x)$ have the following representation in B

$$D^{\varepsilon}(x) = I + \varepsilon D_1(x) + \varepsilon^2 D_2(x) + \varepsilon^2 D_0^{\varepsilon}(x),$$

where operators $D_2(x)$ on B are closed and $B_0 := \cap_{x \in E} \mathcal{D}(D_2(x))$ is dense in B, $\overline{B}_0 = B$; operators $D_0^{\varepsilon}(x)$ are a negligible operator, i.e., $\lim_{\varepsilon \to 0} \|D_0^{\varepsilon}(x)\varphi\| = 0$.

D2: The following balance condition holds

$$\Pi D_1(x) \Pi = 0. \tag{1}$$

D3: The moments $m_3(x), x \in E$, are uniformly bounded.

Diffusion approximation

THEOREM

Under Assumptions A1, A5-A6, and D1-D3, the following weak convergence takes place

$$\Phi^{\varepsilon}_{[t/\varepsilon^2]} \Longrightarrow \Phi_0(t), \quad \varepsilon \to 0,$$

where the limit random evolution $\Phi_0(t)$ is a "diffusion random evolution" determined by the following generator

$$\mathbb{L} = \Pi D_2(x)\Pi + \Pi D_1(x) R_0 D_1(x)\Pi - \Pi D_1^2(x)\Pi.$$

Diffusion approximation with equilibrium

$$W_t^\varepsilon := \varepsilon^{-1/2}[\Phi^\varepsilon_{[t/\varepsilon]} - \overline{\Phi}(t)].$$

Theorem

Under Assumptions A1, A5-A6, and D3, with operators $D^\varepsilon(x)$ in A3, instead of D1, the deviated semi-Markov random evolution W_t^ε weakly convergence, when $\varepsilon \to 0$, to the diffusion random evolution W_t^0 defined by the following generator

$$\mathbb{L} = \Pi(D_1(x) - \overline{D}_1)R_0(D_1(x) - \overline{D}_1)\Pi.$$

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Geometric Markov Renewal Process (GMRP)

We consider GMRP as an example of a DTSMRE.
GMRP is a generalization of Aase (1988) geometric compound Poisson process in finance and Cox-Ross-Rubinstein (1976) discrete time model for the stock price.

Geometric Markov Renewal Process (GMRP)

Let $\rho(x)$ be a bounded continuous function on E such that $\rho(x) > -1$. The following functional on semi-Markov chain

$$S_k := S_0 \prod_{l=1}^{k} (1 + \rho(z_l)), \qquad (3)$$

where $S_0 > 0$, is called *Geometric Markov Renewal Process* (GMRP).

We model stock price as S_k-GMRP.

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The GMRP

If we define the operator $D(z)$ on $C_0(\mathbb{R})$ in the following way

$$D(z)\varphi(s) := \varphi(s(1+\rho(z))),$$

then the discrete-time semi-Markov random evolution Φ_k has the following presentation:

$$\Phi_k\varphi(s) = \prod_{l=1}^{k} D(z_l)\varphi(s) = \varphi(s\prod_{l=1}^{k}(1+\rho(z_l)) = \varphi(S_k),$$

where $S(0) = S_0 = s$, and we suppose that $\prod_{k=1}^{0} = 1$.

Geometric Compound Poisson Process as a GMRP

The GMRP process we call such by analogy with the geometric compound Poisson process

$$S_t = S_0 \prod_{k=1}^{N(t)} (1 + Y_k),$$

where $S_0 > 0$, $N(t)$ is a standard Poisson process, $(Y_k)_{k \in \mathbb{N}}$ are i.i.d. r.v., which is a trading model in many financial applications as a pure jump model (See Aase (1988)).

Cox-Ross-Rubinstein Process for Stock Price as a GMRP

If S_k is the stock price at day k, then

$$S_k = S_0 \prod_{l=1}^{k} (1 + \rho_l),$$

where $\rho_l = a$ with probability $p > 0$, and $\rho_l = b$ with probability $1 - p$, where $-1 < a < r < b$ and $r > 0$ is the interest rate. This is Binomial model for stock price (See Cox-Ross-Rubinstein (1976)).

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Averaging of GMRP

Now, define the following sequence of processes:

$$S_t^{\varepsilon} := S_0 \prod_{l=1}^{[t/\varepsilon]} (1 + \varepsilon a(z_l)), \quad t \in \mathbb{R}_+, \quad S_0 = s.$$

Then under averaging conditions the limit process $\bar{S}_k$ has the following form

$$\bar{S}(t) = S_0 e^{\hat{a}t},$$

where $\hat{a} = \int_E \pi(dz) a(z)$.

Diffusion Approximation of GMRP

If we define the following sequence of processes

$$S^{\varepsilon}(t) := S_0 \prod_{l=1}^{[t/\varepsilon^2]} (1 + \varepsilon a(z_l)), \quad t \in \mathbb{R}_+, \quad S_0 = s,$$

then, in the diffusion approximation scheme, we have the following limit process $S_0(t)$

$$S_0(t) = S_0 e^{-t\hat{a}_2/2} e^{\sigma_a w(t)},$$

where

$$\hat{a}_2 := \int_E \pi(dz) a^2(z),$$

$$\sigma_a^2 := \int_E \pi(dz)[a^2(z)/2 + a(z) R_0 a(z)].$$

Diffusion Approximation with Equilibrium of GMRP

Let us consider the following normalized GMRP:

$$w_t^\varepsilon := \varepsilon^{-1/2}[\ln(S_t^\varepsilon / S_0) - \hat{a}t].$$

It is worth noticing that in finance the expression $\ln(S_t^\varepsilon / S_0)$ represents the log-return of the underlying asset (stock, e.g.) S_t^ε. Then this process converges to the following process σw_t, where

$$\sigma^2 = \int_E \pi(a(z) - \hat{a}) R_0 (a(z) - \hat{a}),$$

and w_t is a standard Wiener process.
In this way, the GMRP S_t^ε may be presented in the following approximated form

$$S_t^\varepsilon \approx S_0 e^{\hat{a}t + \sqrt{\varepsilon}\sigma w_t}.$$

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Scheme of Proofs

- Step 1: Compensating operator convergence

$$\Phi^{\varepsilon}_{[t/\varepsilon^r]}, \quad z_{[t/\varepsilon^r]}, \quad \gamma_{[t/\varepsilon^r]}, \quad t \geq 0, \quad \varepsilon > 0, r = 1, 2.$$

Construction of the compensating operator, $\mathbb{L}^{\varepsilon}$ (main part):

- Averaging

$$\mathbb{L}^{\varepsilon}(x) := \varepsilon^{-1} Q^{\sharp} + P^{\sharp} D_1(x) + P^{\sharp} D_0^{\varepsilon}(x)$$

- Diffusion Approximation

$$\mathbb{L}^{\varepsilon}(x) := \varepsilon^{-2} Q^{\sharp} + \varepsilon^{-1} P^{\sharp} D_1(x) + P^{\sharp} D_2(x) + P^{\sharp} D_0^{\varepsilon}(x)$$

Solving the singular perturbation problem

$$\mathbb{L}^{\varepsilon} \phi^{\varepsilon}(u, x) = \mathbb{L}\phi(u) + \theta^{\varepsilon}.$$

Step 2: Tightness

1. The family of processes $\Phi^{\varepsilon}_{[t/\varepsilon]}\varphi$, $\varphi \in B_0$, as $\varepsilon \to 0$, satisfy the compact containment condition and its limit points belong to $C_B[0,\infty)$.
2. The process

$$M^{\varepsilon}_t := \Phi^{\varepsilon}_{[t/\varepsilon]} - I - \sum_{\ell=0}^{[t/\varepsilon]-1} [P^{\sharp} D^{\varepsilon}(\cdot) - I]\Phi^{\varepsilon}_{\ell},$$

is an $\mathcal{F}_{[t/\varepsilon]}$-martingale.
3. The family $\ell(\sum_{k=0}^{[t/\varepsilon]} \mathbb{E}_{\pi}[\Phi^{\varepsilon}_{k+1}\varphi - \Phi^{\varepsilon}_{k}\varphi \mid \mathcal{F}_k])$ is relatively compact for all $\ell \in \mathbb{B}^*_0$, dual of the space B_0.
4. The family $\ell(M^{\varepsilon}_{[t/\varepsilon]}\varphi)$ is relatively compact for any $\ell \in B^*_0$, and any $\varphi \in B_0$.

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Concluding remarks

- RE General Framework.
- We are studying the reduced random media case.
- Additional results for controlled processes are also obtained.
- Additional applications can be considered.

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