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Ruin with Insurance and Financial Risks Following a Dependent Structure

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[1] This talk is based on recent joint works with Yiqing Chen, and Fei Liu

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A discrete-time risk model

- Consider an insurer in a discrete-time risk model with time horizon n.
- Within each period i, the total premium income is denoted by A_i and the total claim amount plus other daily costs is denoted by B_i. Both A_i and B_i are non-negative random variables.
- Suppose that the insurer positions himself in a stochastic economic environment, which leads to an overall stochastic accumulation factor Z_i over each period i.
- Thus, with the initial wealth $W_0 = x$ the current wealth of the insurer at time n is

$$W_n = x \prod_{j=1}^{n} Z_j + \sum_{i=1}^{n} (A_i - B_i) \prod_{j=i+1}^{n} Z_j.$$

Stochastic present value of aggregate net losses

- Introduce
$$X_i = B_i - A_i, \quad Y_i = Z_i^{-1}, \tag{1}$$
which are respectively interpreted as the net loss and the overall stochastic discount factor over period i.
- We call $\{X_i\}$ insurance risks and call $\{Y_i\}$ financial risks.
- The discounted value of the insurer wealth process at time $n$is

$$\begin{aligned} W_n \left(\prod_{j=1}^{n} Y_j\right) &= \left(x \prod_{j=1}^{n} Z_j + \sum_{i=1}^{n} (A_i - B_i) \prod_{j=i+1}^{n} Z_j\right) \left(\prod_{j=1}^{n} Y_j\right) \\ &= x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \\ &= x - S_n. \end{aligned} \tag{2}$$

- The last sum S_n represents the stochastic present value of aggregate net losses up to time n.

The finite-time ruin probability

The probability of ruin by time n is equal to

$$\begin{aligned}\psi(x;n) &= \Pr\left(\inf_{1\le m\le n} W_m < 0\right)\\ &= \Pr\left(\inf_{1\le m\le n} W_m \prod_{j=1}^{m} Y_j < 0\right)\\ &= \Pr\left(\inf_{1\le m\le n} (x - S_m) < 0\right)\\ &= \Pr\left(\max_{1\le m\le n} S_m > x\right)\end{aligned}$$

Therefore, the finite-time ruin probability is the tail probability of the maximal present value of aggregate net losses.

Stochastic present value of aggregate net losses

- Recall the present value of aggregate net losses defined by (2):

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_i.$$

We shall focus on the asymptotic tail behavior of S_n. Anticipated results have immediate applications to calculating risk measures and ruin probability.

- Note that the two risks X_i and Y_i in the same time period i are controlled by the same economic factors (such as global, national or regional economic growth), or affected by a common external event (such as flood, windstorm, forest fire, earthquake or terrorism). Therefore, they should be strongly dependent on each other.
- We assume that $(X_i, Y_i), i \in \mathbb{N}$, form a sequence of i.i.d. random pairs with a generic random pair (X, Y). The components of (X, Y) are dependent through a copula.

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Copulas

- A Bivariate Copula $C(u, v) : [0, 1]^2 \to [0, 1]$ is a joint cumulative distribution function of a random vector on the unit square $[0, 1]^2$ with uniform marginals.
- Let (X, Y) possess marginal distributions F and G and a bivariate copula $C(u, v)$. By Sklar's (1959) theorem,

$$\Pr(X \leq x, Y \leq y) = C(F(x), G(y)).$$

  In particular, if F and G are continuous, then the copula $C(u, v)$ is unique and is identical to the joint distribution of the uniform variates $F(X)$ and $G(Y)$.
- The monographs of Joe (1997) and Nelsen (2006)offer comprehensive treatment on copulas.

FGM distributions

- A bivariate Farlie-Gumbel-Morgenstern (FGM) distribution function is of the form

$$\Pi(x,y) = F(x)G(y)\left(1+\theta(1-F(x))(1-G(y))\right) \tag{3}$$

where F and G are marginal distributions and $|\theta| \le 1$ is a real number.

- The survival copula is defined as

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v).$$

- For the FGM case, we have
$\hat{C}(u,v) = C(u,v) = uv(1+\theta(1-u)(1-v)), \quad (u,v) \in (0,1)^2$

FGM distributions

The FGM distribution describes an asymptotically independent scenario.

- For every $\theta \in [-1, 1]$, the coefficient of upper tail dependence is

$$\chi = \lim_{u \downarrow 0} \frac{\hat{C}(u, u)}{u} = 0.$$

See Section 5.2 of McNeil et al.(2005) for details of the concepts used here.

- Asymptotically independent random variables may still show different degrees of dependence. In this regard, Coles et al. (1999) proposed to use

$$\hat{\chi} = \lim_{u \downarrow 0} \frac{2 \log u}{\log \hat{C}(u, u)} - 1$$

to measure more subtly the strength of dependence in the asymptotic independence case.

FGM distributions

- we see that $\hat{\chi} = 0$ for $\theta \in (-1, 1]$ while $\hat{\chi} = -1/3$ for $\theta = -1$. This illustrates the essential difference between the cases $-1 < \theta \leq 1$ and $\theta = -1$.
- Chen(2011) derived a general asymptotic formula for $\psi(x; n)$, which is not valid for the case $\theta = -1$.
- It turns out that, for the case $\theta = -1$ the asymptotic behavior of $\psi(x; n)$ is essentially different from that for the case $-1 < \theta \leq 1$.
- Recent related discussions can be found in Jiang and Tang (2011), Yang et al.(2011) and Yang and Wang (2013).

Contents

Prevalence of Rare Events

1992 Hurricane Andrew

- Total $ 16 billion in insured losses
- More than 60 insurance companies became insolvent, according to Muermann (2008, NAAJ)
- CBOT launched Insurance CAT futures contracts in 1992

Prevalence of Rare Events

9/11 Attacks

- Almost 3,000 died
- US stocks lost $1.4 trillion during the week
- By the end of 2002, New York City's GDP estimated to have declined by $27.3 billion

Prevalence of Rare Events

2004 Indian Ocean Earthquake and Tsunami

- Damaged about $15 billion
- Over 230,000 were killed
- Not much insurance loss due to lack of insurance coverage
- A 2012 movie, The Impossible, based on the true story of a Spanish family

Prevalence of Rare Events

2005 Hurricane Katrina

- Damaged $108 billion, costliest one in the US
- Insured loss: $41.1 billion

Figure : 2005 Hurricane Katrina

2008 Sichuan Earthquake

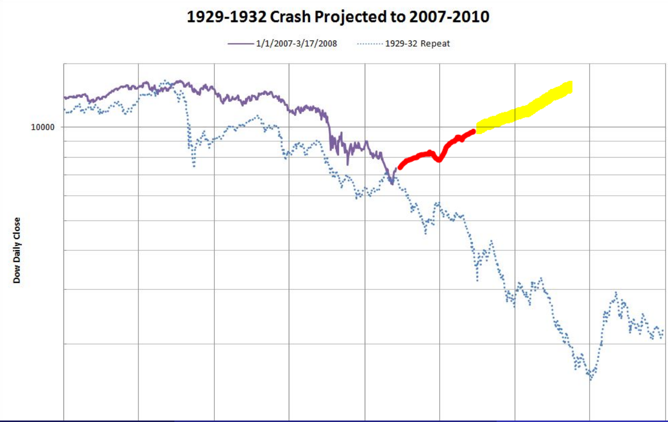
- More than 90,000 died
- Damaged over $20 billion
- Insurers' loss: 1 billion due to not much insurance coverage

Figure : 2008 Sichuan Earthquake

Prevalence of Rare Events

2008 Recession

- Triggered by the collapse of the sub-prime mortgage market in the United States
- X Arguably the worst global recession since the Great Depression in 1930's



Prevalence of Rare Events

2010 Haiti Earthquake

- Killed more than 316,000 people
- Estimated cost: between $7.2–13.2 billion

Prevalence of Rare Events

2011 Japan Earthquake, Tsunami and Nuclear Crisis

- Deaths: Over 16,000
- Insured loss: $14.5-34.6 billion
- World Bank's estimated economic cost: $235 billion

Prevalence of Rare Events

2012 Hurricane Sandy

- Damage: over $68 billion
- Insured loss: $19 billion

Figure : 2010 Haiti Earthquake

Prevalence of Rare Events

2013 Typhoon Haiyan/Yolanda

- Deaths: at least 6,241
- Missing: 1,785
- Damage: $1.5 billion

Long tailed distributions

- A distribution function F on $\mathbb{R}$ is said to be long tailed, written as $F \in \mathcal{L}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and the relation

$$\overline{F}(x+y) \sim \overline{F}(x) \tag{4}$$

  holds for some (or, equivalently, for all) $y \neq 0$.

- For $F \in \mathcal{L}$, automatically there is some real function $l(\cdot)$ with $0 < l(x) \leq x/2$ and $l(x) \uparrow \infty$ such that relation (4) holds uniformly for $y \in [x - l(x), x + l(x)]$; that is,

$$\lim_{x\to\infty} \sup_{x-l(x)\leq y\leq x+l(x)} \left| \frac{\overline{F}(x+y)}{\overline{F}(x)} - 1 \right| = 0.$$

Subexponential distributions

It is well known that $\mathcal{S} \subset \mathcal{L}$

- A distribution function F on $\mathbb{R}^+ = [0, \infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and

$$\overline{F^{2*}}(x) \sim 2\overline{F}(x),$$

  where F^{2*} denotes the two-fold convolution of F.

- More generally, a distribution function F on $\mathbb{R}$ is still said to be subexponential if the distribution function $F_+(x) = F(x)1_{(x \geq 0)}$ is subexponential.

Subclasses of the subexponential class

- A distribution function F on $\mathbb{R}$ is said to be dominatedly-varying tailed, written as $F \in \mathcal{D}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and the relation

$$\overline{F}(xy) = O\left(\overline{F}(x)\right)$$

  holds for some (or, equivalently, for all) $0 < y < 1$.
- The intersection $\mathcal{L} \cap \mathcal{D}$ covers the class $\mathcal{C}$ of distributions with a consistently-varying tail.
- A distribution function F on $\mathbb{R}$, we write $F \in \mathcal{C}$ if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and

$$\lim_{y \downarrow 1} \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

  Clearly, for $F \in \mathcal{C}$ it holds for every $o(x)$ function that $\overline{F}(x + o(x)) \sim \overline{F}(x)$.

Subclasses of the subexponential class

- A distribution function F on $\mathbb{R}$, we write $F \in \mathcal{R}_{-\alpha}$ for some $0 \leq \alpha < \infty$ if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and the relation

$$\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x)$$

holds for all $y > 0$, and we write $\mathcal{R}$ the union of $\mathcal{R}_{-\alpha}$ over $0 \leq \alpha < \infty$.

- An extension of regular variation is rapid variation
A distribution F on $\mathbb{R}$ is said to have a rapidly-varying tail, denoted by $\mathcal{R}_{-\infty}$, if $\overline{F}(x) > 0$ for all x and

$$\lim_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 0$$

holds for all $y > 1$.

- Their relations are as follow: $\mathcal{R} \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$

Matuszewska indices

- For a distribution function F with $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$, it upper and lower Matuszewska indices are defined as

$$J_F^+ = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y} : y > 1\right\}$$

and

$$J_F^- = \sup\left\{-\frac{\log \overline{F}^*(y)}{\log y} : y > 1\right\},$$

where $\overline{F}_*(y) = \liminf \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup \overline{F}(xy)/\overline{F}(x)$.

- It is clear that $F \in \mathcal{D}$ if and only if $0 \leq J_F^+ < \infty$, while if $F \in \mathcal{R}_{-\alpha}$ for $0 \leq \alpha \leq \infty$ then $J_F^+ = J_F^- = \alpha$.

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General assumptions

- We assume that (X_i, Y_i), $i \in \mathbb{N}$, form a sequence of i.i.d. random pairs with a generic random pair (X, Y)
- However, the components of (X, Y) are dependent and follow a joint bivariate FGM distribution.
- Denote by F, G and H the distribution function of X, Y and XY, respectively.
- We introduce independent random variables $X_\vee^*$ and $X_\wedge^*$, independent of all other sources of randomness, with $X_\vee^*$ identically distributed as $X_1^* \vee X_2^*$ and with $X_\wedge^*$ identically distributed as $X_1^* \wedge X_2^*$, where X_1^* and X_2^* are two i.i.d. copies of X.

General assumptions

- Recall that the dependence structure of (X, Y) is described by the joint distribution function (3) with $\theta = -1$; that is

$$\Pi(x, y) = F(x)G(y)\left(1 - \overline{F}(x)\overline{G}(y)\right) \tag{5}$$

with F on $\mathbb{R}$ and G on $\mathbb{R}^+$.

- Introduce independent random variables X^*, Y^*, Y_1^*, Y_2^*, Y_3^*, ..., with the first identically distributed as X and the other identically distributed as Y.

The first main result

In the first result below, the condition $0 < \hat{y} \leq 1$ indicates that there are risk-free investments only:

Theorem

Let the random pair (X, Y) follow a bivariate FGM distribution function (5) with $F \in \mathcal{S}$ and $0 < \hat{y} \leq 1$. Then it holds for each $n \in \mathbb{N}$ that

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x \right), \tag{6}$$

where, and throughout the paper, the usual convention $\prod_{j=2}^{1} Y_j^* = 1$ is in force.

The second main result

In the second result below, the condition $1 \le \hat{y} \le \infty$ allows to include risky investments:

Theorem

Let the random pair (X, Y) follow a bivariate FGM distribution function (5) with $F \in \mathcal{L}$, $1 \le \hat{y} \le \infty$ and $H \in \mathcal{S}$. The relation

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \sum_{i=1}^{n} \Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right) \qquad (7)$$

holds for each $n \in \mathbb{N}$ under either of the following groups of conditions:

(i) *there is an auxiliary function $a(\cdot)$ such that $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$ and $\overline{H}(x - a(x)) \sim \overline{H}(x)$,*

(ii) *$J_F^- > 0$, and there is an auxiliary function $a(\cdot)$ such that $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$.*

The third main result

Theorem

Let the random pair (X, Y) follow a bivariate FGM distribution function (5) with $F \in \mathcal{S}$ and $0 < \hat{y} < \infty$. Then relation

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x \right), \tag{8}$$

holds for each $n \in \mathbb{N}$.

This result extends the first main result by relaxing the restriction on Y from $0 < \hat{y} \leq 1$ to $0 < \hat{y} < \infty$.

The fourth main result

Theorem

Let the random pair (X, Y) *follow a bivariate FGM distribution function (5). Relation*

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x \right), \tag{9}$$

holds for each $n \in \mathbb{N}$ *under either of the following groups of conditions:*

(i) $F \in \mathcal{C}$ *and* $\mathrm{E}[Y^p] < \infty$ *for some* $p > J_F^+$,

(ii) $F \in \mathcal{L} \cap \mathcal{D}$ *with* $J_F^- > 0$, *and* $\mathrm{E}[Y^p] < \infty$ *for some* $p > J_F^+$.

Contents

Further Discussion

- In the first and second main results, if $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$, then applying Breiman's theorem (see Cline and Samorodnitsky (1994), who attributed it to Breiman (1965)) to relation (9), we obtain

$$\psi(x;n) \sim \mathrm{E}\left[(Y_{\wedge}^{*})^{\alpha}\right] \frac{1-\left(\mathrm{E}\left[Y^{\alpha}\right]\right)^{n}}{1-\mathrm{E}\left[Y^{\alpha}\right]} \overline{F}(x), \qquad (10)$$

  where the ratio $\frac{1-(\mathrm{E}[Y^{\alpha}])^{n}}{1-\mathrm{E}[Y^{\alpha}]}$ is understood as n if $\alpha = 0$.
- Relation (10) is identical to relation (3.2) of Chen (2011) with $\theta = -1$.

Corollary I

In the next two corollaries we look at a critical situation with the same heavy-tailed insurance and financial risks. The first one below addresses the regular variation case:

Corollary

Let the random pair (X, Y) *follow a bivariate FGM distribution function (5). If* $F \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 0$, $\overline{F}(x) \sim c\overline{G}(x)$ *for some* $c > 0$, *and* $\mathrm{E}[Y^{\alpha}] = \infty$, *then it holds for each* $n \in \mathbb{N}$ *that*

$$\psi(x; n) \sim \left(c\mathrm{E}\left[(Y_{\wedge}^{*})^{\alpha}\right] + \mathrm{E}\left[\left(X_{\wedge}^{*+}\right)^{\alpha}\right]\right) \Pr\left(\prod_{j=1}^{n} Y_j^{*} > x\right). \tag{11}$$

Corollary II

The second one below addresses the rapid variation case:

Corollary

Let the random pair (X, Y) *follow a bivariate FGM distribution function (5). If* $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ *and* $\overline{F}(x) \sim c\overline{G}(x)$ *for some* $c > 0$, *then it holds for each* $n \in \mathbb{N}$ *that*

$$\psi(x; n) \sim (1 + c) \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{n} Y_j^* > x \right). \qquad (12)$$

Thank you very much for your attention!