On a Convex Measure of Drawdown Risk

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A single large drawdown may force liquidation at the bottom of the market...

... and the ensuing market recovery is never experienced

Figure: Simulation of a fictional portfolio’s net asset value over a finite path.
Going beyond “What’s the Drawdown?”

1. Form expectations about future potential drawdowns

2. Integrate drawdown in the investment process

3. Add value on top of volatility and Expected Shortfall

Outline:

1. Measuring drawdown risk in terms of *Conditional Expected Drawdown (CED)*

2. Properties of CED relevant to the investment process

3. Drawdown risk and serial correlation
Measuring Drawdown Risk

The object of interest is a random vector \( X_{T_n} = (X_{t_1}, \ldots, X_{t_n}) \) representing a finite path of cumulative returns, where \( X_{t_i} = P_{t_i}/P_{t_1} - 1 \), and \( P_{t_i} \) denotes the price level.

The Maximum Drawdown within a path \( X_{T_n} \) is the largest drop from peak to trough within that path:

\[
\mu(X_{T_n}) = \max \max_{1 \leq i < n} \max_{i < j \leq n} \{X_{t_j} - X_{t_i}, 0\}
\]

For \( \alpha \in (0, 1) \), Conditional Expected Drawdown (CED) measures the average of worst case maximum drawdowns exceeding a quantile of the maximum drawdown distribution:

\[
\text{CED}_{\alpha}(X_{T_n}) = TM_{\alpha}(\mu(X_{T_n})) = \frac{1}{1 - \alpha} \int_{\alpha}^{1} DT_u(\mu(X_{T_n})) \, du
\]

\( DT_{\alpha} \), the Maximum Drawdown Threshold, is a quantile of \( \mu(X_{T_n}) \):

\[
DT_{\alpha}(\mu(X_{T_n})) = \inf \{\mu^* \mid \mathbb{P}(\mu(X_{T_n}) > \mu^*) \leq 1 - \alpha\}
\]

It is thus the smallest maximum drawdown \( \mu^* \) such that the probability that the maximum drawdown \( \mu(X_{T_n}) \) exceeds \( \mu^* \) is at most \( (1 - \alpha) \).
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Maximum Drawdown Distributions

- A related measure of drawdown risk developed by Chekhlov et al. (2003, 2005) considers time series of drawdowns.
- Distributions of maximum drawdowns within a path of fixed length incorporate a time horizon parameter that is relevant to the investor.
- Measurements of drawdown risk and the risk of extreme returns are analogous.

Figure: Empirical distribution of returns (L) and 6-month maximum drawdowns (R) for the daily S&P 500 over the period 1 January 1950 to 31 December 2013, together with the corresponding 90% Tail Means (ES and CED).
CED Properties Relevant to the Investment Process: Convexity

In Föllmer and Schied (2002, 2010, 2011), the essence of diversification is encapsulated in the convexity property:

CED\[\alpha\] : \(\mathcal{V}_T \rightarrow \mathbb{R}\) satisfies the convexity axiom; that is for all \(X_T, Y_T \in \mathcal{V}_T\) and \(\lambda \in [0, 1]\),

\[
CED_{\alpha}(\lambda X_T + (1 - \lambda)Y_T) \leq \lambda CED_{\alpha}(X_T) + (1 - \lambda)CED_{\alpha}(Y_T)
\]

CED can be used in an optimizer, and the optimization can be formulated as an efficient linear programming problem.\(^2\)

\(^2\)This formulation uses the LP algorithm of shortfall optimization developed by Rockafellar and Uryasev (2000, 2002).
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CED Properties Relevant to the Investment Process: Positive Homogeneity

**Portfolio return** can be written as a weighted sum of factor returns:

\[ P = \sum_{i=1}^{n} w_i X_i \]

Portfolio risk \( \rho(P) \) is not a weighted sum of individual source risks. But there is a parallel in terms of **marginal risk contributions**:

\[ \sum_{i} w_i \frac{\partial \rho(P)}{\partial w_i} = \rho(P) \]

CED\( \alpha \colon \mathcal{V}_{T_n} \to \mathbb{R} \) satisfies degree-one positive homogeneity; that is for all \( X_{T_n} \in \mathcal{V}_{T_n} \) and \( \lambda > 0 \),

\[ \text{CED}_\alpha(\lambda X_{T_n}) = \lambda \text{CED}_\alpha(X_{T_n}) \]

The overall drawdown risk CED of a portfolio \( P = \sum_{i} w_i X_i \) can be linearly attributed along its factors \( X_i \) (Euler’s Homogenous Function Theorem).

**Remark:** CED is not a coherent risk measure in the sense of Artzner et al. (1999).
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Portfolio risk $\rho(P)$ is not a weighted sum of individual source risks. But there is a parallel in terms of marginal risk contributions:

$$ \sum_{i} w_i \frac{\partial \rho(P)}{\partial w_i} = \rho(P) $$

CED$_\alpha : \mathcal{V}_T \rightarrow \mathbb{R}$ satisfies degree-one positive homogeneity; that is for all $X_T \in \mathcal{V}_T$ and $\lambda > 0$,

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Drawdown is inherently path dependent – in a way, it measures the degree to which losses are sustained.

Empirical studies showing that CED captures temporal dependence to a greater degree than volatility and Expected Shortfall.

The impact of serial correlation on risk is greater for CED than for return-based measures:

Monte Carlo simulation of an autoregressive AR(1) model

\[ r_t = \kappa r_{t-1} + \epsilon_t \]

with varying values for \( \kappa \) and \( \epsilon \) is Gaussian with variance 0.1.
Drawdown Risk and Serial Correlation

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Drawdown Risk and Serial Correlation

The impact of serial correlation on risk is greater for CED than for return-based measures (cont.):

We fit the AR(1) model $r_t = \kappa r_{t-1} + \epsilon_t$ to a daily time series of US Equity and US Government Bonds on a 6-month rolling basis to obtain time series of estimated $\kappa$ values for each asset.

<table>
<thead>
<tr>
<th></th>
<th>Volatility Time Series</th>
<th>ES$_{0.9}$ Time Series</th>
<th>CED$_{0.9}$ Time Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>US Equity $\kappa$ time series</td>
<td>0.47</td>
<td>0.52</td>
<td>0.75</td>
</tr>
<tr>
<td>US Bonds $\kappa$ time series</td>
<td>0.32</td>
<td>0.39</td>
<td>0.69</td>
</tr>
</tbody>
</table>

Table: For the daily time series of each of US Equity and US Government Bonds, correlations of estimates of the autoregressive parameter $\kappa$ in an AR(1) model with the values of the three risk measures (volatility, 90% Expected Shortfall and 90% Conditional Expected Drawdown) estimated over 1982–2013.
Drawdown Risk and Serial Correlation

The impact of serial correlation on risk is greater for CED than for return-based measures (cont.):

**Figure:** For each of US Equity and US Government Bonds, scatter plots of the daily time series of 6-month rolling estimates of the autoregressive parameter $\kappa$ with the 6-month rolling estimates of 90% Conditional Expected Drawdown.

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The value of going beyond “What’s the Drawdown?”

- Value of capturing autocorrelation in practice
- Forecasting drawdown risk
- Optimizing drawdown risk
- Drawdown mitigating strategies
Thank You!
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