Tail risk measures

For

Generalized Skew Elliptical Family

Tomer Shushi, Zinoviy Landsman, Udi Makov

University of Haifa,

ISRAEL

8th Conference in Actuarial Science & Finance on Samos

• In this paper we investigate a risk measures called *tail conditional expectation* (TCE), *tail variance* (TV) and *tail variance-to-mean ratio* (TVMR), which are well studied in the Multivariate Normal Family for essentially more wide Elliptical Family.

• It is well known that loss data and returns may have non-symmetric distributions.

We provide the TCE, TV and TVMR conceptions for Generalized Skew Elliptical (GSE) family of distributed risks.

The tail conditional expectation

Notations:

X : loss random variable

$F_X(x)$: distribution function

$\bar{F}_X(x) = 1 - F_X(x)$: tail function

x_q : q-th quantile

$$x_q =: VaR_q(X)$$

The tail conditional expectation

$$TCE_q(X) := E(X \mid X > x_q)$$

is interpreted as the expected worst possible loss.

$$TCE_q(X) = x_q + E(X - x_q \mid X > x_q)$$

$$\geq VaR_q(X)$$

The tail variance (TV)

Furman and Landsman (2006).

$$TV_q(X) := Var \quad X \mid X > x_q$$

$$= E(\quad X - TCE_q(X) \quad ^2 \mid X > x_q),$$

is interpreted as the variance of the worst possible loss.

The tail variance-to-mean ratio (TVMR)

$$TVMR_q(X) := \frac{TV_q(X)}{TCE_q(X)},$$

This measure provides us a tool for examine
The normalized measure of the tail variance
under some quantile q, and by that examine the
dispersion of extreme values of X.

The calculation of TCE has some prehistory. Panjer (1999) obtained TCE and TCE-based allocation for the multivariate normal family of distributions.

Landsman and Valdez (2003) generalized the previous results for the elliptical family of distributions.

We extend these results for the class of GSE distributions.

The class of GSE distributions

$$f_{\mathbf{Y}}(\mathbf{y}) =$$

$$= \frac{2}{\sqrt{|\Sigma|}} g^{(n)}\left(\frac{1}{2}\ \mathbf{y}-\mu\ ^{T}\Sigma^{-1}\ \mathbf{y}-\mu\ \right) H\ \ \gamma^{T}\left[\Sigma^{-1/2}\ \mathbf{y}-\mu\ \right].$$

Azzalini and Capitanio (2003)

We say

$$\mathbf{Y} \sim GSE_{n}\ \ \mu, \Sigma, \gamma, g^{(n)}, H$$

(a) for $\gamma > 0,$ the distribution has a right tail

(b) for $\gamma = 0,$ the distribution is symmetric

(elliptical distribution)

(c) for $\gamma < 0,$ the distribution has a left tail

Multivariate Skew Normal distribution

Azzalini and Dalla Valle (1996)

$$\mathbf{Y} \sim SN_n \quad \mu, \Sigma, \gamma$$

$$g^{(n)}(x) = \frac{1}{(2\pi)^{n/2}} \exp(-x), \; H(x) = \Phi(x),$$

So

$$f_{\mathbf{Y}}(\mathbf{y}) =$$

$$= \frac{2}{(2\pi)^{n/2}\sqrt{|\Sigma|}} \exp\left(-\left(\frac{1}{2} (\mathbf{y}-\mu)^T \Sigma^{-1} (\mathbf{y}-\mu) \right)\right) \Phi\left(\gamma^T \left[\Sigma^{-1/2} (\mathbf{y}-\mu) \right] \right)$$

The impact of the shape parameter

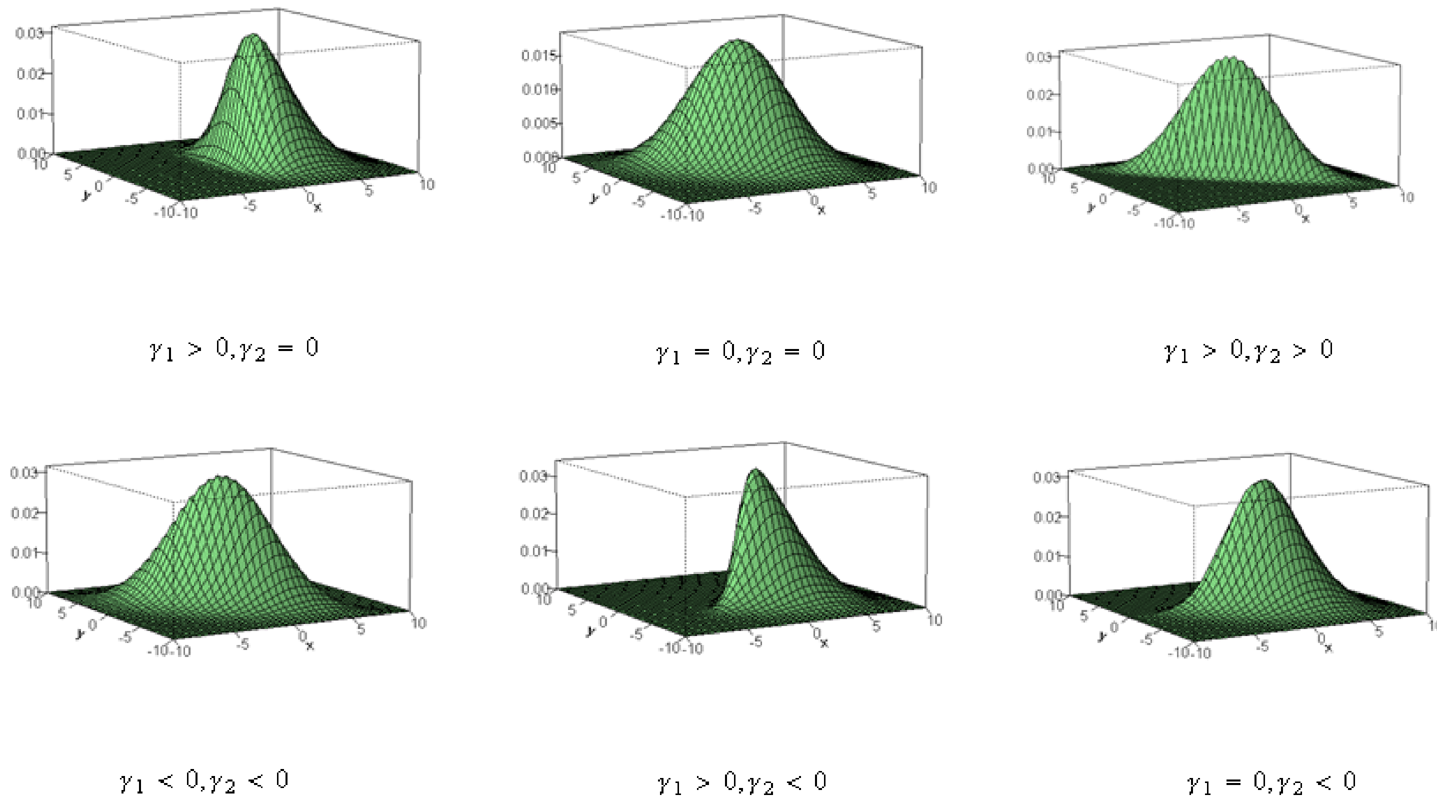

Figure 1: Comparing Bivariate *(X,Y)* Densities for Skew Normal distributions with the shape parameters γ_1, γ_2.

TCE for univariate GSE distributions

Define

$$\overline{G}(z) := \int_{z}^{\infty} g^{(1)}(u)du,$$

Which is called the cumulative generator.

Theorem 1

Under condition $\overline{G}(0) < \infty$

the TCE for an univariate GSE distribution is given by

$$TCE_q \;\; Y \;\; = \mu + \Lambda_q \sigma.$$

where

$$\Lambda_q := 2 \cdot \frac{\overline{G}\left(\frac{1}{2}\zeta_q^2\right) H \;\; \gamma\zeta_q \;\; + \gamma k \;\; \zeta_q}{1-q},$$

$$\zeta_q = VaR_q \quad \zeta \quad = \frac{y_q - \mu}{\sigma},$$

and

$$k \quad \zeta_q \quad := \int_{\varsigma_q}^{\infty} \overline{G}\left(\frac{1}{2}t^2\right) \cdot h(\gamma t)dt. \qquad (2)$$

Skew Normal distribution

$$f_Y(y) = \frac{2}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)\Phi\left(\gamma\frac{y-\mu}{\sigma}\right),$$

Then

$$\Lambda_q = \frac{2}{1-q}\left(\Phi\;\;\gamma\zeta_q\;\;\cdot\phi\;\;\zeta_q\;\;+\frac{\gamma}{\sqrt{2\pi}\sqrt{\gamma^2+1}}\overline{\Phi}\;\;\sqrt{\gamma^2+1}\zeta_q\;\;\right)$$

Which conforms with Vernic (2006).

Skew Student-t

$$Y \sim SSt_1\left(\mu, \sigma^2, \gamma, m\right),$$

when the pdf of *Y* is given by

$$f_Y(y) = \frac{2}{\sigma} t_m\left(\frac{y-\mu}{\sigma}\right) T_m\left(\gamma \frac{y-\mu}{\sigma}\right).$$

Here t_m and T_m are pdf and cdf of the standard student-t distribution with *m* degrees of freedom.

$$TCE_q \quad Y \quad = \mu + 2 \cdot \frac{\overline{G}\left(\frac{1}{2}\zeta_q^2\right) T_m \quad \gamma\zeta_q \quad + \gamma k \quad \zeta_q}{1-q}\sigma,$$

Where

$$\overline{G}\left(\frac{1}{2}\zeta_q^2\right) = \sqrt{\frac{m}{m-2}}\, t_{m-2}\left(\sqrt{\frac{m-2}{m}}\zeta_q\right),$$

and

$$k \;\; \zeta_q \;\; =$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\pi\Gamma\left(\frac{m}{2}\right)}\left[\eta - \zeta_q F_1\left(\frac{1}{2};\frac{m-1}{2},\frac{m+1}{2};\frac{3}{2};-\frac{\zeta_q^2}{m},-\frac{\gamma\zeta_q \quad {}^2}{m}\right)\right],$$

Where $F_1 \;\; a, b_1, b_2, c; x, y$ is Appell hypergeometric function of two variables (*x* and *y*).

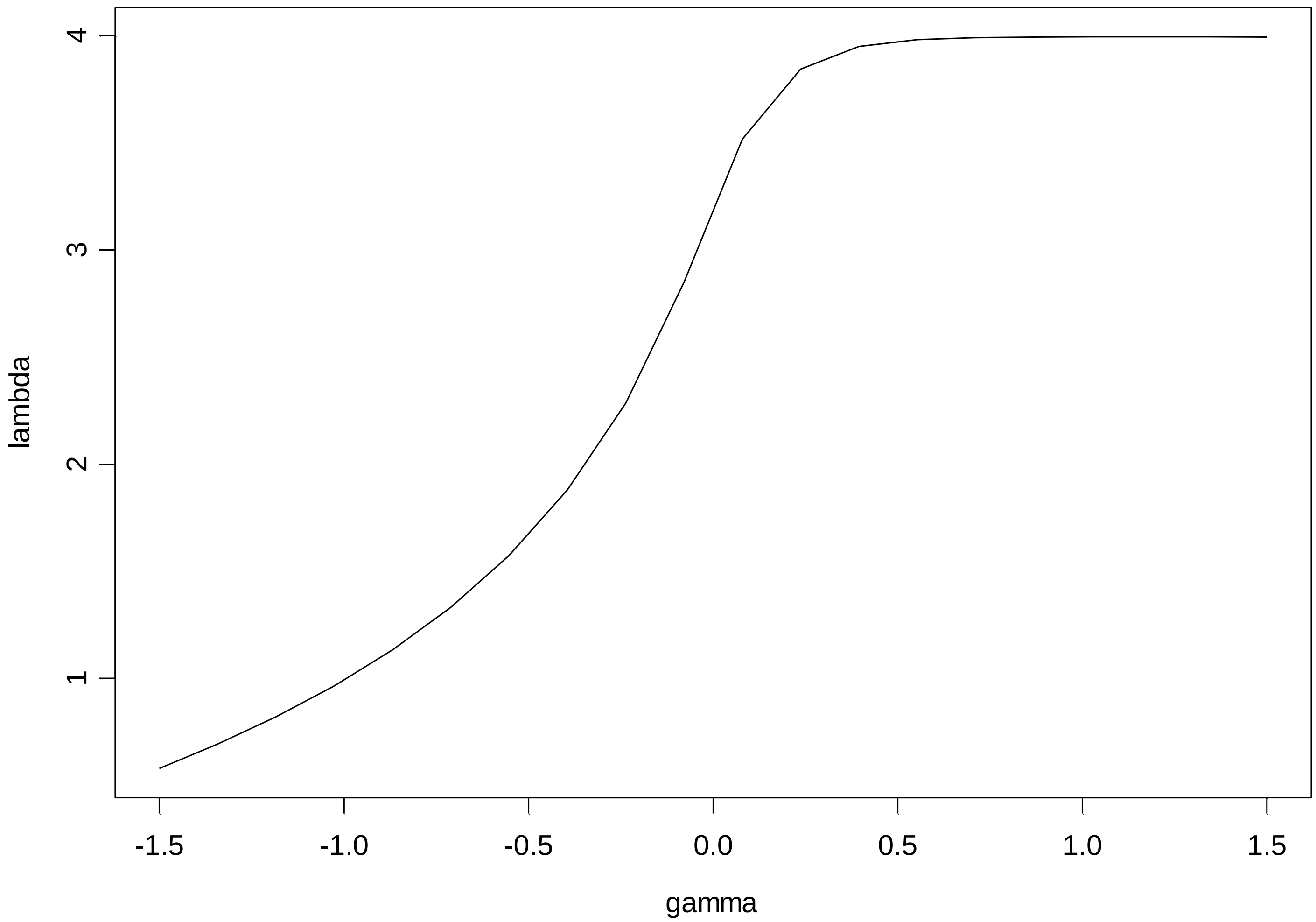

Figure 2: Graph of the relation between γ and Λ_q for standard SSt

Skew Normal-Laplace

$$Y \sim SN - L_1\left(\mu, \sigma^2, \gamma\right),$$

with the pdf

$$f_Y(y) = \begin{cases} \dfrac{1}{\sigma}\phi\left(\dfrac{y-\mu}{\sigma}\right)e^{\gamma\frac{y-\mu}{\sigma}}, & \dfrac{y-\mu}{\sigma} \le 0 \\ 2\dfrac{1}{\sigma}\phi\left(\dfrac{y-\mu}{\sigma}\right)\left(1-\dfrac{1}{2}e^{-\gamma\frac{y-\mu}{\sigma}}\right), & \dfrac{y-\mu}{\sigma} \ge 0 \end{cases},$$

Nadarajah and Kotz (2003). Recall

$$TCE_q(Y) = \mu + \Lambda_q \sigma,$$

Where

$$\Lambda_q = \frac{1}{1-q}\left(2\phi(\zeta_q) - e^{\frac{1}{2}\gamma^2}\phi(\zeta_q + \gamma) - \gamma\overline{\Phi}(\zeta_q + \gamma)\right)$$

TCE for weighted sum of GSE distributions

Suppose

$$\mathbf{X} \sim GSE_k\left(\mu, \Sigma, \gamma, g^{(k)}, H\right),$$

$$R = \mathrm{x}^{\mathbf{T}}\mathbf{X} \sim GSE_1\left(\mathrm{x}^{\mathbf{T}}\mu, \mathrm{x}^{\mathbf{T}}\Sigma\mathrm{x}, \tilde{\gamma}, g^{(1)}, H\right).$$

So

$$TCE_q\left(R\right) = \mathrm{x}^T\mu + \Lambda_q\sqrt{\mathrm{x}^T\Sigma\mathrm{x}}$$

TV for univariate GSE distributions

Define

$$\overline{G}_{g,h}(u) := \frac{1}{\sigma^2{}_Z} \int_u^{\infty} \overline{G}(t) h(\sqrt{2t}\gamma^2) dt < \infty, \quad u \geq 0,$$

And let ζ^* be an associated with ζ random variable.

$$\overline{F}_{\zeta^*}(u) := \int_u^{\infty} f_{\zeta^*}(u) dt = 2 \int_u^{\infty} f_Z(t) H(\gamma t) dt,$$

Theorem 2

Under conditions

$$\overline{G}(0) < \infty$$

and

$$\sigma^2{}_Z < \infty$$

The TV for an univariate GSE distribution is given by

$$TV_q\left(Y\right) = 2\sigma^2\sigma_Z{}^2\left[r_1\left(\zeta_q,\gamma\right) - r_2\left(\zeta_q,\gamma\right)\right],$$

Where

$$r_1 \;\; \zeta_q , \gamma \;\; := \frac{1}{1-q}\left(H \;\; \gamma\zeta_q \;\;\; f_{Z^*} \;\; \zeta_q \;\; \zeta_q + \gamma \overline{G}_{g,h}\left(\frac{1}{2}\zeta_q{}^2 \right) + \frac{1}{2}\overline{F}_{\varsigma^*} \;\; \zeta_q \;\;\; \right),$$

and

$$r_2 \;\; \zeta_q , \gamma \;\; := 2\left(\frac{H \;\; \gamma\zeta_q \;\;\; f_{Z^*} \;\; \zeta_q \;\; + \gamma\kappa \;\; \zeta_q}{1-q} \right)^2 .$$

Skew Normal distribution

Recall

$$f_Y(y) = \frac{2}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)\Phi\left(\gamma\frac{y-\mu}{\sigma}\right),$$

and

$$\Lambda_q = \frac{2}{1-q}\left(\Phi\ \ \gamma\zeta_q\ \ \cdot\phi\ \ \zeta_q\ \ + \frac{\gamma}{\sqrt{2\pi}\sqrt{\gamma^2+1}}\overline{\Phi}\ \ \sqrt{\gamma^2+1}\zeta_q\ \ \right).$$

Then

$$r_1(\zeta_q,\gamma) = \frac{1}{1-q}\left(\Phi(\gamma\zeta_q)\,\varphi(\zeta_q)\,\zeta_q + \frac{\gamma\varphi(\zeta_q)\sqrt{1+\gamma^2}}{\sqrt{2\pi}\,(1+\gamma^2)} + \frac{1}{2}\overline{F}_{\zeta^*}(\zeta_q) \right),$$

$$r_2(\zeta_q,\gamma) = 2\left(\frac{\Phi(\gamma\zeta_q)\,\varphi(\zeta_q) + \frac{\gamma\varphi(\zeta_q)\sqrt{1+\gamma^2}}{\sqrt{2\pi}\sqrt{1+\gamma^2}}\,\overline{\Phi}\left(\sqrt{1+\gamma^2}\,\zeta_q\right)}{1-q} \right)^2,$$

Where $F_{\zeta^*}(\zeta_q)$ is the cdf of skew normal distribution.

Thank you!