Tail risk measures
For
Generalized Skew Elliptical Family

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• In this paper we investigate a risk measures called *tail conditional expectation* (TCE), *tail variance* (TV) and *tail variance-to-mean ratio* (TVMR), which are well studied in the Multivariate Normal Family for essentially more wide Elliptical Family.

• It is well known that loss data and returns may have non-symmetric distributions.

We provide the TCE, TV and TVMR conceptions for Generalized Skew Elliptical (GSE) family of distributed risks.
The tail conditional expectation

Notations:

\( X \) : loss random variable

\( F_X(x) \) : distribution function

\( \bar{F}_X(x) = 1 - F_X(x) \) : tail function

\( x_q \) : \( q \)-th quantile

\( x_q =: \text{VaR}_q(X) \)
The tail conditional expectation

$$TCE_q(X) := E(X \mid X > x_q)$$

is interpreted as the expected worst possible loss.

$$TCE_q(X) = x_q + E(X - x_q \mid X > x_q) \geq VaR_q(X)$$
The tail variance (TV) is interpreted as the variance of the worst possible loss.

\[ TV_q(X) := \text{Var} \ X \mid X > x_q \]

\[ = E \left( X - TCE_q(X)^2 \right) \mid X > x_q, \]

is interpreted as the variance of the worst possible loss.
The tail variance-to-mean ratio (TVMR)

\[ TVMR_q(X) := \frac{TV_q(X)}{TCE_q(X)}, \]

This measure provides us a tool for examine
The normalized measure of the tail variance
under some quantile \( q \), and by that examine the
dispersion of extreme values of \( X \).
The calculation of TCE has some prehistory. Panjer (1999) obtained TCE and TCE-based allocation for the multivariate normal family of distributions. Landsman and Valdez (2003) generalized the previous results for the elliptical family of distributions. We extend these results for the class of GSE distributions.
The class of GSE distributions

\[ f_Y(y) = \]

\[ = \frac{2}{\sqrt{|\Sigma|}} g^{(n)} \left( \frac{1}{2} y - \mu \right)^T \Sigma^{-1} (y - \mu) H \gamma^T \left[ \Sigma^{-1/2} (y - \mu) \right]. \]

We say

\[ Y \sim GSE_n \mu, \Sigma, \gamma, g^{(n)}, H \]

Azzalini and Capitanio (2003)
(a) for $\gamma > 0$, the distribution has a right tail

(b) for $\gamma = 0$, the distribution is symmetric (elliptical distribution)

(c) for $\gamma < 0$, the distribution has a left tail
Multivariate Skew Normal distribution

Azzalini and Dalla Valle (1996)

\[ Y \sim SN_n \mu, \Sigma, \gamma \]

\[ g^{(n)}(x) = \frac{1}{(2\pi)^{n/2}} \exp(-x), \quad H(x) = \Phi(x), \]
So

\[
    f_y(y) = \frac{2}{(2\pi)^{n/2} \sqrt{|\Sigma|}} \exp \left( -\left( \frac{1}{2} \begin{bmatrix} y - \mu \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} y - \mu \end{bmatrix} \right) \right) \Phi \gamma^T \begin{bmatrix} \Sigma^{-1/2} & y - \mu \end{bmatrix}
\]

The impact of the shape parameter
Figure 1: Comparing Bivariate \((X,Y)\) Densities for Skew Normal distributions with the shape parameters \(\gamma_1, \gamma_2\).
TCE for univariate GSE distributions

Define

\[ \bar{G}(z) := \int_{z}^{\infty} g^{(1)}(u) du, \]

Which is called the cumulative generator.
Theorem 1

Under condition $\overline{G}(0) < \infty$

the TCE for an univariate GSE distribution is given by

$$TCE_q Y = \mu + \Lambda_q \sigma.$$ 

where

$$\Lambda_q := 2 \cdot \frac{\overline{G} \left( \frac{1}{2} \zeta_q^2 \right) H \gamma \zeta_q + \gamma k \zeta_q}{1 - q}.$$
\[ \zeta_q = \text{VaR}_q \quad \zeta = \frac{y_q - \mu}{\sigma}, \]

and

\[ k \zeta_q := \int_{\zeta_q}^{\infty} G\left(\frac{1}{2}t^2\right) \cdot h(\gamma t) dt. \quad (2) \]
Skew Normal distribution

\[ f_Y(y) = \frac{2}{\sigma} \phi\left(\frac{y - \mu}{\sigma}\right) \Phi\left(\gamma \frac{y - \mu}{\sigma}\right), \]

Then

\[ \Lambda_q = \frac{2}{1 - q} \left( \Phi \gamma \zeta_q \cdot \phi \zeta_q + \frac{\gamma}{\sqrt{2\pi} \sqrt{\gamma^2 + 1}} \overline{\Phi} \sqrt{\gamma^2 + 1} \zeta_q \right) \]

Skew Student-t

\[ Y \sim SS_{t_1} \mu, \sigma^2, \gamma, m \],

when the pdf of \( Y \) is given by

\[
f_Y(y) = \frac{2}{\sigma} t_m \left( \frac{y - \mu}{\sigma} \right) T_m \left( \gamma \frac{y - \mu}{\sigma} \right).
\]

Here \( t_m \) and \( T_m \) are pdf and cdf of the standard student-t distribution with \( m \) degrees of freedom.
$TCE_q \ Y = \mu + 2 \cdot \frac{\bar{G}\left(\frac{1}{2} \zeta_q^2\right) T_m \gamma \zeta_q + \gamma k \zeta_q}{1 - q} \sigma,$

Where

$$\bar{G}\left(\frac{1}{2} \zeta_q^2\right) = \sqrt{\frac{m}{m-2}} t_{m-2} \left(\sqrt{\frac{m-2}{m}} \zeta_q\right),$$
and

\[
\begin{align*}
    k \quad \zeta_q &= \\
    &= \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\pi\Gamma\left(\frac{m}{2}\right)} \left[ \eta - \zeta_q \, F_1 \left( \frac{1}{2}; \frac{m-1}{2}, \frac{m+1}{2}; \frac{3}{2}; -\frac{\zeta_q^2}{m}, -\frac{\gamma\zeta_q^2}{m} \right) \right],
\end{align*}
\]

Where \( F_1 \ a, b_1, b_2, c; x, y \) is Appell hypergeometric function of two variables \( (x \text{ and } y) \).
Figure 2: Graph of the relation between $\gamma$ and $\Lambda_q$ for standard SST
Skew Normal-Laplace

\[ Y \sim SN - L_1 \mu, \sigma^2, \gamma, \]

with the pdf

\[
f_Y(y) = \begin{cases} 
\frac{1}{\sigma} \phi\left( \frac{y - \mu}{\sigma} \right) e^{\frac{y - \mu}{\sigma}}, & \frac{y - \mu}{\sigma} \leq 0 \\
2 \frac{1}{\sigma} \phi\left( \frac{y - \mu}{\sigma} \right) \left(1 - \frac{1}{2} e^{-\frac{y - \mu}{\sigma}}\right), & \frac{y - \mu}{\sigma} \geq 0 
\end{cases}
\]

Nadarajah and Kotz (2003). Recall
\[ TCE_q \ Y = \mu + \Lambda_q \sigma, \]

Where

\[ \Lambda_q = \frac{1}{1-q} \begin{pmatrix} 2\phi & \zeta_q & -e^{\frac{1}{2}\nu^2} & \phi & \zeta_q + \gamma & -\gamma \Phi \end{pmatrix} \]
\[ \zeta_q + \gamma \]
TCE for weighted sum of GSE distributions

Suppose

\[ X \sim GSE_k \mu, \Sigma, \gamma, g^{(k)}, H, \]

\[ R = x^T X \sim GSE_1 x^T \mu, x^T \Sigma x, \tilde{\gamma}, g^{(1)}, H. \]

So

\[ TCE_q R = x^T \mu + \Lambda_q \sqrt{x^T \Sigma x} \]
TV for univariate GSE distributions

Define

\[ \bar{G}_{g,h}(u) := \frac{1}{\sigma^2} \int_{u}^{\infty} G(t)h(\sqrt{2t\gamma^2}) dt < \infty, \quad u \geq 0, \]

And let \( \zeta^* \) be an associated with \( \zeta \) random variable.

\[ \bar{F}_{\zeta^*}(u) := \int_{u}^{\infty} f_{\zeta^*}(u) dt = 2 \int_{u}^{\infty} f_Z(t)H(\gamma t) dt, \]
Theorem 2

Under conditions

\[ \bar{G}(0) < \infty \]

and

\[ \sigma^2_Z < \infty \]

The TV for an univariate GSE distribution is given by

\[ TV_q\ Y = 2\sigma^2\sigma_Z^2 \left[ r_1 \zeta_q, \gamma - r_2 \zeta_q, \gamma \right], \]
Where

\[ r_1 \xi_q, \gamma := \frac{1}{1-q} \left( H \gamma \xi_q f_{z*} \xi_q \xi_q + \gamma G_{g,h} \left( \frac{1}{2} \xi_q^2 \right) + \frac{1}{2} \overline{F_{\xi*}} \xi_q \right), \]

and

\[ r_2 \xi_q, \gamma := 2 \left( \frac{H \gamma \xi_q f_{z*} \xi_q + \gamma \kappa \xi_q}{1-q} \right)^2. \]
Skew Normal distribution

Recall

\[ f_Y(y) = \frac{2}{\sigma} \phi \left( \frac{y - \mu}{\sigma} \right) \Phi \left( \gamma \frac{y - \mu}{\sigma} \right), \]

and

\[ \Lambda_q = \frac{2}{1-q} \left( \Phi \left( \gamma \zeta_q \right) \cdot \phi \left( \zeta_q \right) + \frac{\gamma}{\sqrt{2\pi} \sqrt{\gamma^2 + 1}} \Phi \left( \sqrt{\gamma^2 + 1} \zeta_q \right) \right). \]
Then
\[
\begin{align*}
    r_1 \xi_q, \gamma &= \frac{1}{1-q} \left( \Phi \gamma \xi_q \varphi \xi_q + \frac{\gamma \varphi \xi_q \sqrt{1+\gamma^2}}{\sqrt{2\pi} (1+\gamma^2)} + \frac{1}{2} F_{\xi^*} \xi_q \right), \\
    r_2 \xi_q, \gamma &= 2 \left( \Phi \gamma \xi_q \varphi \xi_q + \frac{\gamma \varphi \xi_q \sqrt{1+\gamma^2}}{\sqrt{2\pi} \sqrt{1+\gamma^2}} \Phi \sqrt{1+\gamma^2} \xi_q \right)^2.
\end{align*}
\]

Where \( F_{\xi^*} \xi_q \) is the cdf of skew normal distribution.
Thank you!