Pricing Currency Derivatives with Markov-modulated Lévy Dynamics

Anatoliy Swishchuk

University of Calgary, Canada

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Abstract
Markov-modulated Lévy Processes
Markov-Modulated Lévy Dynamics: Main Results
Example: Double Exponential Distribution
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This talk introduces dynamic models for the spot foreign exchange rate with capturing both the rare events and the time-inhomogeneity in the fluctuating currency market. For the rare events, we use a Lévy process, and for the time-inhomogeneity in the market dynamics, we indicate the strong dependence of the domestic/foreign interest rates, the appreciation rate and the volatility of the foreign currency on the time-varying sovereign ratings in the currency market. The time-varying ratings are formulated by a continuous-time finite-state Markov chain.
Abstract

We study the pricing of some currency options adopting a so-called regime-switching Esscher transform to identify a risk-neutral martingale measure. By determining the regime-switching Esscher parameters we then get an integral expression on the prices of European-style currency options. Finally, numerical illustrations are presented as well.
Finite state Markov chain

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a probability measure $P$. Consider a continuous-time, finite-state Markov chain $\xi = \{\xi_t\}_{0 \leq t \leq T}$ on $(\Omega, \mathcal{F}, P)$ with a state space $\mathcal{S}$, the set of unit vectors $(e_1, \cdots, e_n) \in \mathbb{R}^n$ with a rate matrix $\Pi$. The dynamics of the chain are given by

$$\xi_t = \xi_0 + \int_0^t \Pi \xi_u du + M_t \in \mathbb{R}^n,$$

where $M = \{M_t, t \geq 0\}$ is a $\mathbb{R}^n$-valued martingale with respect to $(\mathcal{F}_t^\xi)_{0 \leq t \leq T}$, the $P$-augmentation of the natural filtration, generated by the Markov chain $\xi$. 

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Modeling a Spot FX Rate

A Markov-modulated Lévy dynamics, which models the dynamics of the spot FX rate, is given by the following SDE:

\[ dS_t = S_{t-} \left( \mu_t dt + \sigma_t dW_t + (e^{Z_t} - 1) dN_t \right). \]

Here \( \mu_t \) is drift parameter; \( W_t \) is a Brownian motion, \( \sigma_t \) is the volatility; \( N_t \) is a Poisson Process with intensity \( \lambda_t \), the jump size is controlled by \( Z_t \). The distribution of \( Z_t \) has a density \( \nu(x), x \in \mathbb{R} \). All sources of randomness are independent.
The parameters $\mu_t$, $\sigma_t$, $\lambda_t$ are modeled using the finite state Markov chain

\[
\begin{align*}
\mu_t &:= < \mu, \xi_t >, \mu \in \mathbb{R}^n; \\
\sigma_t &:= < \sigma, \xi_t >, \sigma \in \mathbb{R}_+^n; \\
\lambda_t &:= < \lambda, \xi_t >, \lambda \in \mathbb{R}_+^n.
\end{align*}
\]
Solution of SDE Running the Dynamics of FX Rate

The solution of (2) is \( S_t = S_0 e^{L_t} \), (where \( S_0 \) is the spot FX rate at time \( t = 0 \)). Here \( L_t \) is given by the formula

\[
L_t = \int_0^t (\mu_s - 1/2\sigma_s^2)ds + \int_0^t \sigma_s dW_s + \int_0^t Z_s dN_s.
\]
Discounted Spot FX Rate

Domestic and foreign interest rates \((r^d_t)_{0 \leq t \leq T}, (r^f_t)_{0 \leq t \leq T}\) are defined also by finite state Markov chain \((\xi_t)_{0 \leq t \leq T}\):

\[
  r^d_t = \langle r^d, \xi_t \rangle, r^d \in \mathbb{R}^n_+,
\]
\[
  r^f_t = \langle r^f, \xi_t \rangle, r^f \in \mathbb{R}^n_+.
\]

Discounted spot FX rate:

\[
  S^D_t = \exp \left( \int_0^t (r^f_s - r^d_s) ds \right) S_t, \; 0 \leq t \leq T. \tag{5}
\]
Using (2), Itô’s formula with jumps we derive SDE for discounted spot FX rate

\[
dS^D_t = S^D_t (r^d_t - r^f_t + \mu_t)dt + S^D_t \sigma_t dW_t + S^D_t (e^{Z_t} - 1)dN_t.
\]
Log Spot FX Rate

Log spot FX rate

\[ Y_t = \log \left( \frac{S_t^D}{S_0} \right) \]

Using Itô’s formula with jumps

\[ Y_t = C_t + J_t, \]

where \( C_t, J_t \) are continuous and jump parts of \( Y_t \).

\[ C_t = \int_0^t \left( r^d_s - r^f_s + \mu_s \right) ds + \int_0^t \sigma_s dW_s, \tag{7} \]

\[ J_t = \int_0^t Z_s^- dN_s \tag{8} \]
Let \((\mathcal{F}_t^Y)_{0 \leq t \leq T}\) denote the \(\mathbb{P}\)-augmentation of the natural filtration, generated by \(Y\). For each \(t \in [0, T]\) set \(\mathcal{H}_t = \mathcal{F}_t^Y \vee \mathcal{F}_T^\xi\). Let us also define two families of regime switching parameters \((\theta^c_s)_{0 \leq s \leq T}, (\theta^J_s)_{0 \leq s \leq T}\):

\[
\theta^m_t = \langle \theta^m, \xi_t \rangle,
\]

\[
\theta^m = (\theta^m_1, \ldots, \theta^m_n) \subset \mathbb{R}^n,
\]

\[
m = \{c, J\}.
\]
Esscher Transform

Define a random Esscher transform $\mathbb{Q}^{\theta^c, \theta^J} \sim \mathbb{P}$ on $\mathcal{H}_t$ using these families of parameters $(\theta^c_s)_{0 \leq s \leq T}$, $(\theta^J_s)_{0 \leq s \leq T}$

$$L^{\theta^c, \theta^J}_t = \left. \frac{d\mathbb{Q}^{\theta^c, \theta^J}}{d\mathbb{P}} \right|_{\mathcal{H}_t} =: \frac{\exp \left( \int_0^t \theta^c_s dC_s + \int_0^t \theta^J_s dJ_s \right)}{\mathbb{E} \left[ \exp \left( \int_0^t \theta^c_s dC_s + \int_0^t \theta^J_s dJ_s \right) \mid \mathcal{F}_t \right]}. \quad (9)$$
The density $L_{t}^{\theta_{c},\theta_{J}}$ of Esscher transform defined in (9) is:

$$L_{t}^{\theta_{c},\theta_{J}} = \exp\left(\int_{0}^{t} \theta_{c}^{c} \sigma_{s} dW_{s} - 1/2 \int_{0}^{t} (\theta_{c}^{c} \sigma_{s})^2 ds\right) \times$$

$$\exp\left(\int_{0}^{t} \theta_{J}^{J} Z_{s} dN_{s} - \int_{0}^{t} \lambda_{s} \left(\int_{\mathbb{R}} e^{\theta_{J}^{J} x} \nu(dx) - 1\right) ds\right).$$

In addition, the random Esscher transform density $L_{t}^{\theta_{c},\theta_{J}}$ is an exponential $(\mathcal{H}_{t})_{0 \leq t \leq T}$ martingale and satisfies the following SDE:

$$\frac{dL_{t}^{\theta_{c},\theta_{J}}}{L_{t-}^{\theta_{c},\theta_{J}}} = \theta_{c}^{c} \sigma_{t} dW_{t} + (e^{\theta_{J}^{J} Z_{t} - 1}) dN_{t} - \lambda_{t} \left(\int_{\mathbb{R}} e^{\theta_{J}^{J} x} \nu(dx) - 1\right) dt.$$
Martingale Condition for Discounted Spot FX Rate

Martingale condition for discounted spot FX rate $S_t^D$

$$\mathbb{E}^{\theta_c,\theta^J}[S_t^D | \mathcal{H}_u] = S_u^D, \quad t \geq u. \quad (11)$$

To derive such a condition Bayes formula is used

$$\mathbb{E}^{\theta_c,\theta^J}[S_t^D | \mathcal{H}_u] = \frac{\mathbb{E}[L_t^{\theta_c,\theta^J} S_t^D | \mathcal{H}_u]}{\mathbb{E}[L_t^{\theta_c,\theta^J} | \mathcal{H}_u]} = \mathbb{E}\left[\frac{L_t^{\theta_c,\theta^J}}{L_u^{\theta_c,\theta^J}} S_t^D | \mathcal{H}_u\right] \quad (12)$$
Theorem

Let the random Esscher transform be defined by (9). Then the martingale condition (for $S^D_t$, see (12)) holds if and only if the Markov modulated parameters $(\theta^c_t, \theta^J_t, 0 \leq t \leq T)$ satisfy for all $0 \leq t \leq T$ the condition

$$r^f_t - r^d_t + \mu_t + \theta^c_t \sigma^2_t + \lambda^\theta_t^J k^\theta_t^J = 0. \tag{13}$$

Here the random Esscher transform intensity $\lambda^\theta_t^J$ of the Poisson Process and the mean percentage jump size $k^\theta_t^J$ are, respectively, given by

$$\lambda^\theta_t^J = \lambda_t \int_{\mathbb{R}} e^{\theta^J_s x} \nu(dx), \quad k^\theta_t^J = \frac{\int_{\mathbb{R}} e^{(\theta^J_t + 1) x} \nu(dx)}{\int_{\mathbb{R}} e^{\theta^J_t x} \nu(dx)} - 1, \tag{14}$$

as long as $\int_{\mathbb{R}} e^{\theta^J_t x} \nu(dx) < +\infty$, $\int_{\mathbb{R}} e^{(\theta^J_t + 1) x} \nu(dx) < +\infty$. 

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The new density of jumps $\tilde{\nu}$ is defined by the following formula

$$
\frac{\int_{\mathbb{R}} e^{(\theta_t J + 1)x} \nu(dx)}{\int_{\mathbb{R}} e^{\theta_t J x} \nu(dx)} = \int_{\mathbb{R}} e^{x \tilde{\nu}}(dx).
$$

(15)
Regime-switching parameters satisfying martingale condition for spot FX rate

\begin{align}
\theta_{t}^{c, *} &= \frac{r_{t}^{d} - r_{t}^{f} - \mu_{t}}{\sigma_{t}^{2}}, \quad (16) \\
\theta_{t}^{J, *} : \frac{\int_{\mathbb{R}} e^{(\theta_{t}^{J, *} + 1)x} \nu(dx)}{\int_{\mathbb{R}} e^{\theta_{t}^{J, *} x} \nu(dx)} &= 1. \quad (17)
\end{align}
Double Exponential Distribution

It is defined by the following formula of the density function

\[ \nu(x) = p\theta_1 e^{-\theta_1 x} \mathbb{1}_{x \geq 0} + (1 - p)\theta_2 e^{\theta_2 x} \mathbb{1}_{x < 0}, \]  

(18)

where \( \theta_1 > 1, \quad \theta_2 > 0. \)

The mean value of this distribution is

\[ \text{mean}(\theta_1, \theta_2, p) = \frac{p}{\theta_1} - \frac{1 - p}{\theta_2}. \]  

(19)

The variance of this distribution is

\[ \text{var}(\theta_1, \theta_2, p) = \frac{2p}{\theta_1^2} + \frac{2(1 - p)}{\theta_2^2} - \left( \frac{p}{\theta_1} - \frac{1 - p}{\theta_2} \right)^2. \]  

(20)
Regime-switching Parameters Satisfying Martingale Condition for Spot FX Rate

The family of regime switching Esscher transform parameters is defined by (16), (17). The parameter $\theta^J_t$, (the first parameter $\theta^C_t$ has the same formula as in general case) is defined by (see (15))

$$\int_{\mathbb{R}} e^{(\theta^J_t + 1)x} \left( p \theta_1 e^{-\theta_1 x} \left| x \geq 0 \right. \right) + (1 - p) \theta_2 e^{\theta_2 x} \left| x < 0 \right. \right) dx = \left(21\right)$$

$$\int_{\mathbb{R}} e^{\theta^J_t x} \left( p \theta_1 e^{-\theta_1 x} \left| x \geq 0 \right. \right) + (1 - p) \theta_2 e^{\theta_2 x} \left| x < 0 \right. \right) dx.$$
Regime-switching Parameters Satisfying Martingale Condition for Spot FX Rate

We require an additional restriction for the convergence of the integrals in (21)

\[-\theta_2 < \theta^J_t < \theta_1.\]  

(22)

If \(p\theta_1 - (1 - p)\theta_2 \neq 0\) we have two solutions and one of them satisfies restriction (22)

\[
\theta^J_t = -\frac{p\theta_1 + 2\theta_1 \theta_2 - (1 - p)\theta_2}{2(p\theta_1 - (1 - p)\theta_2)} \pm \]

(23)

\[
((p\theta_1 + 2\theta_1 \theta_2 - (1 - p)\theta_2)^2 - 4(p\theta_1 - (1 - p)\theta_2)(p\theta_1 \theta_2(\theta_1 + \theta_2) - \theta_2 \theta_1^2 + \theta_1 \theta_2))^{0.5}(2(p\theta_1 - (1 - p)\theta_2))^{-1}.
\]
The New Poisson Process Intensity and the New Mean Jump Size

Then the Poisson process intensity is

$$\lambda_{t}^{\theta,J} = \lambda_{t} \left( \frac{p\theta_{1}}{\theta_{1} - \theta_{t}} + \frac{(1-p)\theta_{2}}{\theta_{2} + \theta_{t}} \right).$$  \hspace{1cm} (24)

The new mean jump size is

$$k_{t}^{\theta,J} = 0$$  \hspace{1cm} (25)
The New Distribution of Jumps

If we proceed to a new risk-neutral measure $Q$ we have a new density of jumps $\nu$

$$\tilde{\nu}(x) = \tilde{p}\theta_1 e^{-\theta_1 x} I_{x \geq 0} + (1 - \tilde{p})\theta_2 e^{\theta_2 x} I_{x < 0}. \quad (26)$$
The New Distribution of Jumps

\[ \tilde{p} = \frac{p\theta_1}{\theta_1 - \theta_i^J - 1} + \frac{(1-p)\theta_2}{\theta_2 + \theta_i^J + 1} - \frac{\theta_2}{\theta_2 + 1} \cdot \frac{\theta_1}{\theta_1 - 1} - \frac{\theta_2}{\theta_2 + 1} \]  

(27)
We now proceed to the general formulas for European calls (see Merton (1976)). For the European call currency options with a strike price $K$ and the time of expiration $T$ the price at time zero is given by

$$\Pi_0(S, K, T, \xi) = \mathbb{E}^{\theta^c_*, \theta^J_*} \left[ e^{-\int_0^T (r^d_s - r^f_s) ds} (S_T - K)^+ | \mathcal{F}_T^\xi \right],$$

(28)

where the spot FX rate dynamics $S_T$ is considered under the equivalent domestic martingale measure.
Valuation of European Style Currency Options

Let $J_i(t,T)$ denote the occupation time of $\xi$ in state $e_i$ over the period $[t,T], t < T$. We introduce several new quantities that will be used in future calculations

$$R_{t,T} = \frac{1}{T-t} \int_t^T (r^d_s - r^f_s) ds = \frac{1}{T-t} \sum_{i=1}^n (r^d_i - r^f_i) J_i(t,T),$$  \hspace{1cm} (29)$$

$$U_{t,T} = \frac{1}{T-t} \int_t^T \sigma^2_s ds = \frac{1}{T-t} \sum_{i=1}^n \sigma^2_i J_i(t,T),$$  \hspace{1cm} (30)$$

$$\lambda^{\theta \star J}_{t,T} = \frac{1}{T-t} \sum_{i=1}^n \lambda^{\theta \star J}_i J_i(t,T),$$  \hspace{1cm} (31)$$

$$\lambda^{\theta \star \star}_t = \frac{1}{T-t} \int_t^T (1 + k^\theta_s J) \lambda^{\theta \star J}_s ds = \frac{1}{T-t} \sum_{i=1}^n (1 + k^\theta_i J) \lambda^{\theta \star J}_i J_i(t,T),$$  \hspace{1cm} (32)$$
Valuation of European Style Currency Options

\[ V_{t,T,m}^2 = U_{t,T} + \frac{m\sigma_J^2}{T - t}, \]  

\[ R_{t,T,m} = R_{t,T} - \frac{1}{T - t} \int_t^T \lambda_s^\theta^* J k_s^\theta^* J ds + \frac{1}{T - t} \int_t^T \frac{\log(1 + k_s^\theta^* J)}{T - t} ds = \]  

\[ R_{t,T} - \frac{1}{T - t} \sum_{i=1}^n \lambda_i^\theta^* J k_i^\theta^* J + \frac{m}{T - t} \sum_{i=1}^n \frac{\log(1 + k_i^\theta^* J)}{T - t} J_i(t,T), \]  

where \( J_i(t,T) := \int_t^T < \xi_s, e_i > ds, \) \( \sigma_J^2 \) is the variance of the distribution of the jumps, \( m \) is the number of jumps in the interval \([t, T]\), \( n \) is the number of states of the Markov chain \( \xi \).
From the pricing formula in Merton (1976) let us define

$$\Pi_0(S, K, T; R_{0,T}, U_{0,T}, \lambda_{0,T}) = \sum_{m=0}^{\infty} \frac{e^{-T\lambda_{0,T}^*J} \left(T\lambda_{0,T}^*\right)^m}{m!} \times (35)$$

$$BS_0(S, K, T, V_{0,T,m}^2, R_{0,T,m}),$$

where $BS_0(S, K, T, V_{0,T,m}^2, R_{0,T,m})$ is the standard Black-Scholes price formula with initial spot FX rate $S$, strike price $K$, risk-free rate $r$, volatility square $\sigma^2$ and time $T$ to maturity.
The European style call option pricing formula takes the form:

\[
\Pi_0(S, K, T) = \int_{[0,t]^n} \Pi_0(S, K, T; R_{0,T}, U_{0,T}, \lambda_{0,T}^{\Theta^*, J}) \times \psi(J_1, J_2, ..., J_n) dJ_1 ... dJ_n.
\]

Here, \(\psi(J_1, J_2, ..., J_n)\) is the joint probability distribution density for the occupation time \(J_i(t, T) \equiv \int_t^T < \xi_s, e_i > ds,\).
Numerical Simulations

- In the Figures 1-6 we shall provide numerical simulations for the case when the amplitude of jumps is described by the double exponential distribution.
- These graphs show a dependence of the European-call option price against $S/K$, where $S$ is the initial spot FX rate, $K$ is the strike FX rate for a different maturity time $T$ in years: 0.5, 1, 1.2.
- Blue line denotes the log-double exponential, green line denotes the log-normal, red-line denotes the plot without jumps.
Numerical Simulations

Figure: 1 $S_0 = 1, T = 0.5, \theta_1 = 10, \theta_2 = 10, p = 0.5$, mean normal = 0, sigma normal = 0.1
Figure: $S_0 = 1, T = 1.0, \theta_1 = 10, \theta_2 = 10, p = 0.5$, mean normal = 0, sigma normal = 0.1.
Numerical Simulations

Figure: \( S_0 = 1, T = 1.2, \theta_1 = 10, \theta_2 = 10, p = 0.5, \) mean normal = 0, sigma normal = 0.1
Numerical Simulations

Figure: \( S_0 = 1, T = 0.5, \theta_1 = 5, \theta_2 = 10, p = 0.5, \) mean normal = 0, sigma normal = 0.1
Numerical Simulations

Figure: 5 $S_0 = 1, T = 1.0, \theta_1 = 5, \theta_2 = 10, p = 0.5, \text{mean normal } = 0, \text{sigma normal } = 0.1$
Numerical Simulations

**Figure**: $S_0 = 1, T = 1.2, \theta_1 = 5, \theta_2 = 10, p = 0.5, \text{mean normal} = 0, \sigma_{\text{normal}} = 0.1$
Numerical Simulations

If we fix the value of the $\theta_2$ parameter in the double exponential distribution with $S/K = 1$ the corresponding plot is given in Figure 7.
Bo et al. (2010) Results

This research generalizes the results by L. Bo, Y. Wang, X. Yang 'Markov-modulated jump-diffusion for currency option pricing' published in *Insurance: Mathematics & Economics*, 2010. The jump process here was modeled as a compound Poisson process with log-normal amplitude to describe the jumps.
Conclusions

- We generalized the formulas of Bo et al. (2010) for a general Lévy process
- We applied obtained formulas to the case of the double exponential distribution of jump size
- We also provided numerical simulations of European call foreign exchange option prices for different parameters

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Thank You for Your Attention and Time!