Pricing Currency Derivatives with Markov-modulated Lévy Dynamics

Anatoliy Swishchuk

University of Calgary, Canada

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Outline

Abstract

This talk introduces dynamic models for the spot foreign exchange rate with capturing both the rare events and the time-inhomogeneity in the fluctuating currency market. For the rare events, we use a Lévy process, and for the time-inhomogeneity in the market dynamics, we indicate the strong dependence of the domestic/foreign interest rates, the appreciation rate and the volatility of the foreign currency on the time-varying sovereign ratings in the currency market. The time-varying ratings are formulated by a continuous-time finite-state Markov chain.

Abstract

We study the pricing of some currency options adopting a so-called regime-switching Esscher transform to identify a risk-neutral martingale measure. By determining the regime-switching Esscher parameters we then get an integral expression on the prices of European-style currency options. Finally, numerical illustrations are presented as well.

Finite state Markov chain

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a probability measure $\mathbf{P}$. Consider a continuous-time, finite-state Markov chain $\xi = \{\xi_t\}_{0 \leq t \leq T}$ on $(\Omega, \mathcal{F}, \mathbf{P})$ with a state space $\mathcal{S}$, the set of unit vectors $(e_1, \cdots, e_n) \in \mathbb{R}^n$ with a rate matrix Π. The dynamics of the chain are given by

$$\xi_t = \xi_0 + \int_0^t \Pi \xi_u du + M_t \in \mathbb{R}^n, \tag{1}$$

where $M = \{M_t, t \geq 0\}$ is a $\mathbb{R}^n$-valued martingale with respect to $(\mathcal{F}_t^{\xi})_{0 \leq t \leq T}$, the $\mathbf{P}$-augmentation of the natural filtration, generated by the Markov chain ξ.

Modeling a Spot FX Rate

A Markov-modulated Lévy dynamics, which models the dynamics of the spot FX rate, is given by the following SDE:

$$dS_t = S_{t_-}\left(\mu_t dt + \sigma_t dW_t + (e^{Z_{t-}} - 1)dN_t\right). \qquad (2)$$

Here μ_t is drift parameter; W_t is a Brownian motion, σ_t is the volatility; N_t is a Poisson Process with intensity λ_t, the jump size is controlled by Z_t. The distribution of Z_t has a density $\nu(x), x \in \mathbb{R}$. All sources of randomness are independent.

Modeling Parameters of the Lévy Process

The parameters μ_t, σ_t, λ_t are modeled using the finite state Markov chain

$$\begin{aligned} \mu_t &:=< \mu, \xi_t >, \ \mu \in \mathbb{R}^n; \\ \sigma_t &:=< \sigma, \xi_t >, \ \sigma \in \mathbb{R}^n_+; \\ \lambda_t &:=< \lambda, \xi_t >, \ \lambda \in \mathbb{R}^n_+. \end{aligned} \tag{3}$$

Solution of SDE Running the Dynamics of FX Rate

The solution of (2) is $S_t = S_0 e^{L_t}$, (where S_0 is the spot FX rate at time $t = 0$). Here L_t is given by the formula

$$L_t = \int_0^t (\mu_s - 1/2\sigma_s^2)ds + \int_0^t \sigma_s dW_s + \int_0^t Z_{s\_} dN_s. \tag{4}$$

Discounted Spot FX Rate

Domestic and foreign interest rates $(r_t^d)_{0\le t\le T}$, $(r_t^f)_{0\le t\le T}$ are defined also by finite state Markov chain $(\xi_t)_{0\le t\le T}$:

$$r_t^d = \langle r^d, \xi_t\rangle, r^d \in \mathbb{R}_+^n,$$

$$r_t^f = \langle r^f, \xi_t\rangle, r^f \in \mathbb{R}_+^n.$$

Discounted spot FX rate:

$$S_t^D = \exp\left(\int_0^t (r_s^f - r_s^d)ds\right) S_t, 0 \le t \le T. \qquad (5)$$

SDE for Discounted Spot FX Rate

Using (2), It$\hat{o}$'s formula with jumps we derive SDE for discounted spot FX rate

$$dS^D_{t_-} = S^D_{t_-}(r^d_t - r^f_t + \mu_t)dt + S^D_{t_-}\sigma_t dW_t + S^D_{t_-}(e^{Z_{t-}} - 1)dN_t. \quad (6)$$

Log Spot FX Rate

Log spot FX rate

$$Y_t = \log\left(\frac{S_t^D}{S_0}\right)$$

Using Itô's formula with jumps

$$Y_t = C_t + J_t,$$

where C_t, J_t are continuous and jump parts of Y_t.

$$C_t = \int_0^t \left(r_s^d - r_s^f + \mu_s\right) ds + \int_0^t \sigma_s dW_s, \tag{7}$$

$$J_t = \int_0^t Z_{s\_} dN_s \tag{8}$$

Esscher Transform

Let $(\mathcal{F}_t^Y)_{0\leq t\leq T}$ denote the **P**-augmentation of the natural filtration, generated by Y. For each $t\in[0,T]$ set $\mathcal{H}_t=\mathcal{F}_t^Y\vee\mathcal{F}_T^\xi$. Let us also define two families of regime switching parameters $(\theta_s^c)_{0\leq s\leq T}$, $(\theta_s^J)_{0\leq s\leq T}$

$$\theta_t^m =< \theta^m, \xi_t >,$$

$$\theta^m = (\theta_1^m, ..., \theta_n^m) \subset \mathbb{R}^n,$$

$$m = \{c, J\}.$$

Esscher Transform

Define a random Esscher transform $\mathbf{Q}^{\theta^c,\theta^J} \sim \mathbf{P}$ on $\mathcal{H}_t$ using these families of parameters $(\theta^c_s)_{0\le s\le T}$, $(\theta^J_s)_{0\le s\le T}$

$$L_t^{\theta^c,\theta^J} = \frac{d\mathbf{Q}^{\theta^c,\theta^J}}{d\mathbf{P}}\bigg|_{\mathcal{H}_t} =: \frac{\exp\left(\int_0^t \theta^c_s dC_s + \int_0^t \theta^J_{s_-} dJ_s\right)}{\mathbb{E}\left[\exp\left(\int_0^t \theta^c_s dC_s + \int_0^t \theta^J_{s_-} dJ_s\right)\bigg|\mathcal{F}^\xi_t\right]}. \qquad (9)$$

Esscher Transform Density

Theorem

The density $L_t^{\theta^c,\theta^J}$ of Esscher transform defined in (9) *is:*

$$L_t^{\theta^c,\theta^J} = \exp\left(\int_0^t \theta_s^c \sigma_s dW_s - 1/2\int_0^t (\theta_s^c\sigma_s)^2 ds\right) \times \quad (10)$$

$$\exp\left(\int_0^t \theta_{s_-}^J Z_{s_-} dN_s - \int_0^t \lambda_s\left(\int_{\mathbb{R}} e^{\theta_s^J x}\nu(dx) - 1\right) ds\right).$$

In addition,the random Esscher transform density $L_t^{\theta^c,\theta^J}$ is an exponential $(\mathcal{H}_t)_{0\le t\le T}$ martingale and satisfies the following SDE

$$\frac{dL_t^{\theta^c,\theta^J}}{L_{t_-}^{\theta^c,\theta^J}} = \theta_t^c\sigma_t dW_t + (e^{\theta_{t_-}^J Z_{t_-}} - 1)dN_t - \lambda_t\left(\int_{\mathbb{R}} e^{\theta_t^J x}\nu(dx) - 1\right) dt.$$

Martingale Condition for Discounted Spot FX Rate

Martingale condition for discounted spot FX rate S_t^D

$$\mathbb{E}^{\theta^c,\theta^J}[S_t^D|\mathcal{H}_u] = S_u^D, \quad t \geq u. \tag{11}$$

To derive such a condition Bayes formula is used

$$\mathbb{E}^{\theta^c,\theta^J}[S_t^D|\mathcal{H}_u] = \frac{\mathbb{E}[L_t^{\theta^c,\theta^J} S_t^D|\mathcal{H}_u]}{\mathbb{E}[L_t^{\theta^c,\theta^J}|\mathcal{H}_u]} = \mathbb{E}\left[\frac{L_t^{\theta^c,\theta^J}}{L_u^{\theta^c,\theta^J}} S_t^D \middle| \mathcal{H}_u\right] \tag{12}$$

Martingale Condition for Discounted Spot FX Rate

Theorem

Let the random Esscher transform be defined by (9). Then the martingale condition(for S_t^D, see (12)) holds if and only if the Markov modulated parameters $(\theta_t^c, \theta_t^J, 0 \le t \le T)$ satisfy for all $0 \le t \le T$ the condition

$$r_t^f - r_t^d + \mu_t + \theta_t^c \sigma_t^2 + \lambda_t^{\theta,J} k_t^{\theta,J} = 0. \tag{13}$$

Here the random Esscher transform intensity $\lambda_t^{\theta,J}$ of the Poisson Process and the mean percentage jump size $k_t^{\theta,J}$ are, respectively, given by

$$\lambda_t^{\theta,J} = \lambda_t \int_{\mathbb{R}} e^{\theta_s^J x} \nu(dx), \quad k_t^{\theta,J} = \frac{\int_{\mathbb{R}} e^{(\theta_t^J+1)x} \nu(dx)}{\int_{\mathbb{R}} e^{\theta_t^J x} \nu(dx)} - 1, \tag{14}$$

as long as $\int_{\mathbb{R}} e^{\theta_t^J x} \nu(dx) < +\infty$, $\int_{\mathbb{R}} e^{(\theta_t^J+1)x} \nu(dx) < +\infty$.

The New Density of Jumps

The new density of jumps $\tilde{\nu}$ is defined by the following formula

$$\frac{\int_{\mathbb{R}} e^{(\theta_t^J+1)x}\nu(dx)}{\int_{\mathbb{R}} e^{\theta_t^J x}\nu(dx)} = \int_{\mathbb{R}} e^x \tilde{\nu}(dx). \tag{15}$$

Regime-switching parameters satisfying martingale condition for spot FX rate

$$\theta_t^{c,*} = \frac{r_t^d - r_t^f - \mu_t}{\sigma_t^2}, \tag{16}$$

$$\theta_t^{J,*} : \frac{\int_{\mathbb{R}} e^{(\theta_t^{J,*}+1)x}\nu(dx)}{\int_{\mathbb{R}} e^{\theta_t^{J,*}x}\nu(dx)} = 1. \tag{17}$$

Double Exponential Distribution

It is defined by the following formula of the density function

$$\nu(x) = p\theta_1 e^{-\theta_1 x}\Big|_{x\geq 0} + (1-p)\theta_2 e^{\theta_2 x}\Big|_{x<0}, \tag{18}$$

where $\theta_1 > 1, \quad \theta_2 > 0.$
The mean value of this distribution is

$$\text{mean}(\theta_1, \theta_2, p) = \frac{p}{\theta_1} - \frac{1-p}{\theta_2}. \tag{19}$$

The variance of this distribution is

$$\text{var}(\theta_1, \theta_2, p) = \frac{2p}{\theta_1^2} + \frac{2(1-p)}{\theta_2^2} - \left(\frac{p}{\theta_1} - \frac{1-p}{\theta_2}\right)^2. \tag{20}$$

Regime-switching Parameters Satisfying Martingale Condition for Spot FX Rate

The family of regime switching Esscher transform parameters is defined by (16), (17). The parameter $\theta_t^{J,*}$, (the first parameter $\theta_t^{c,*}$ has the same formula as in general case) is defined by (see (15))

$$\int_{\mathbb{R}} e^{(\theta_t^J+1)x}\left(p\theta_1 e^{-\theta_1 x}\Big|_{x\geq 0} + (1-p)\theta_2 e^{\theta_2 x}\Big|_{x<0}\right)dx = \tag{21}$$

$$\int_{\mathbb{R}} e^{\theta_t^J x}\left(p\theta_1 e^{-\theta_1 x}\Big|_{x\geq 0} + (1-p)\theta_2 e^{\theta_2 x}\Big|_{x<0}\right)dx.$$

Regime-switching Parameters Satisfying Martingale Condition for Spot FX Rate

We require an additional restriction for the convergence of the integrals in (21)

$$-\theta_2 < \theta_t^J < \theta_1. \tag{22}$$

If $p\theta_1 - (1-p)\theta_2 \neq 0$ we have two solutions and one of them satisfies restriction (22)

$$\theta_t^J = -\frac{p\theta_1 + 2\theta_1\theta_2 - (1-p)\theta_2}{2(p\theta_1 - (1-p)\theta_2)} \pm$$

$$((p\theta_1 + 2\theta_1\theta_2 - (1-p)\theta_2)^2 - 4(p\theta_1 - (1-p)\theta_2)(p\theta_1\theta_2(\theta_1+\theta_2) - \theta_2\theta_1^2 + \theta_1\theta_2))^{0.5}(2(p\theta_1 - (1-p)\theta_2))^{-1}. \tag{23}$$

The New Poisson Process Intensity and the New Mean Jump Size

Then the Poisson process intensity is

$$\lambda_t^{\theta,J} = \lambda_t \left(\frac{p\theta_1}{\theta_1 - \theta_t^J} + \frac{(1-p)\theta_2}{\theta_2 + \theta_t^J} \right). \tag{24}$$

The new mean jump size is

$$k_t^{\theta,J} = 0 \tag{25}$$

The New Distribution of Jumps

If we proceed to a new risk-neutral measure Q we have a new density of jumps ν

$$\tilde{\nu}(x) = \tilde{p}\theta_1 e^{-\theta_1 x}\Big|_{x \geq 0} + (1-\tilde{p})\theta_2 e^{\theta_2 x}\Big|_{x<0}. \tag{26}$$

The New Distribution of Jumps

$$\tilde{p} = \frac{\dfrac{\frac{p\theta_1}{\theta_1 - \theta_t^J - 1} + \frac{(1-p)\theta_2}{\theta_2 + \theta_t^J + 1}}{\frac{p\theta_1}{\theta_1 - \theta_t^J} + \frac{(1-p)\theta_2}{\theta_2 + \theta_t^J}} - \dfrac{\theta_2}{\theta_2 + 1}}{\dfrac{\theta_1}{\theta_1 - 1} - \dfrac{\theta_2}{\theta_2 + 1}}. \tag{27}$$

Valuation of European Style Currency Options

We now proceed to the general formulas for European calls (see Merton (1976)). For the European call currency options with a strike price K and the time of expiration T the price at time zero is given by

$$\Pi_0(S,K,T,\xi) = \mathbb{E}^{\theta^{c,*},\theta^{J,*}}\left[e^{-\int_0^T (r_s^d - r_s^f)ds}(S_T - K)^+ \mid \mathcal{F}_T^{\xi}\right], \qquad (28)$$

where the spot FX rate dynamics S_T is considered under the equivalent domestic martingale measure.

Valuation of European Style Currency Options

Let $J_i(t,T)$ denote the occupation time of ξ in state e_i over the period $[t,T], t < T$. We introduce several new quantities that will be used in future calculations

$$R_{t,T} = \frac{1}{T-t}\int_t^T (r_s^d - r_s^f)ds = \frac{1}{T-t}\sum_{i=1}^n (r_i^d - r_i^f) J_i(t,T), \tag{29}$$

$$U_{t,T} = \frac{1}{T-t}\int_t^T \sigma_s^2 ds = \frac{1}{T-t}\sum_{i=1}^n \sigma_i^2 J_i(t,T), \tag{30}$$

$$\lambda_{t,T}^{\theta^* J} = \frac{1}{T-t}\sum_{i=1}^n \lambda_i^{\theta^* J} J_i(t,T), \tag{31}$$

$$\lambda_{t,T}^{\theta^*} = \frac{1}{T-t}\int_t^T (1+k_s^{\theta^* J})\lambda_s^{\theta^* J} ds = \frac{1}{T-t}\sum_{i=1}^n (1+k_i^{\theta^* J})\lambda_i^{\theta^* J} J_i(t,T), \tag{32}$$

Valuation of European Style Currency Options

$$V_{t,T,m}^2 = U_{t,T} + \frac{m\sigma_J^2}{T-t}, \tag{33}$$

$$R_{t,T,m} = R_{t,T} - \frac{1}{T-t}\int_t^T \lambda_s^{\theta^* J} k_s^{\theta^* J} ds + \frac{1}{T-t}\int_t^T \frac{\log(1+k_s^{\theta^* J})}{T-t} ds = \tag{34}$$

$$R_{t,T} - \frac{1}{T-t}\sum_{i=1}^n \lambda_i^{\theta^* J} k_i^{\theta^* J} + \frac{m}{T-t}\sum_{i=1}^n \frac{\log(1+k_i^{\theta^* J})}{T-t} J_i(t,T),$$

where $J_i(t,T) := \int_t^T < \xi_s,\ e_i > ds$, σ_J^2 is the variance of the distribution of the jumps, m is the number of jumps in the interval $[t,T]$, n is the number of states of the Markov chain ξ.

Valuation of European Style Currency FX Options

From the pricing formula in Merton (1976) let us define

$$\overline{\Pi_0}(S,K,T;R_{0,T},U_{0,T},\lambda_{0,T}^{\theta^*}) = \sum_{m=0}^{\infty} \frac{e^{-T\lambda_{0,T}^{\theta^*,J}}(T\lambda_{0,T}^{\theta^*})^m}{m!} \times \tag{35}$$

$$BS_0(S,K,T,V_{0,T,m}^2,R_{0,T,m}),$$

where $BS_0(S,K,T,V_{0,T,m}^2,R_{0,T,m})$ is the standard Black-Scholes price formula with initial spot FX rate S, strike price K, risk-free rate r, volatility square σ^2 and time T to maturity.

Valuation of European Style Currency FX Options

The European style call option pricing formula takes the form:

$$\Pi_0(S,K,T) = \int_{[0,t]^n} \overline{\Pi_0}(S,K,T;R_{0,T},U_{0,T},\lambda_{0,T}^{\Theta^*,J})\times \quad (36)$$

$$\psi(J_1,J_2,...,J_n)dJ_1...dJ_n.$$

Here, $\psi(J_1,J_2,...,J_n)$ is the joint probability distribution density for the occupation time $J_i(t,T) := \int_t^T < \xi_s,\ e_i > ds,$.

Numerical Simulations

- In the Figures 1-6 we shall provide numerical simulations for the case when the amplitude of jumps is described by the double exponential distribution
- These graphs show a dependence of the European-call option price against S/K, where S is the initial spot FX rate, K is the strike FX rate for a different maturity time T in years: 0.5, 1, 1.2
- Blue line denotes the log-double exponential, green line denotes the log-normal, red-line denotes the plot without jumps

Numerical Simulations

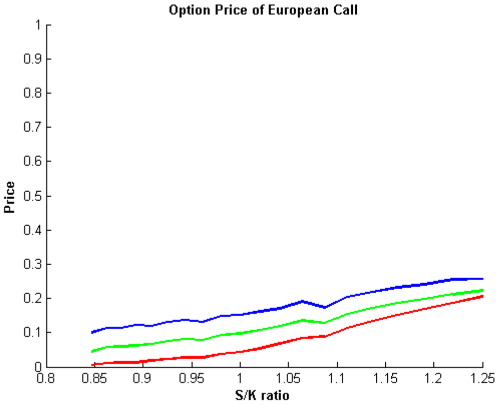


Figure: 1 $S_0 = 1, T = 0.5, \theta_1 = 10, \theta_2 = 10, p = 0.5,$ mean normal $=$ $0,$ sigma normal $= 0.1$

Numerical Simulations

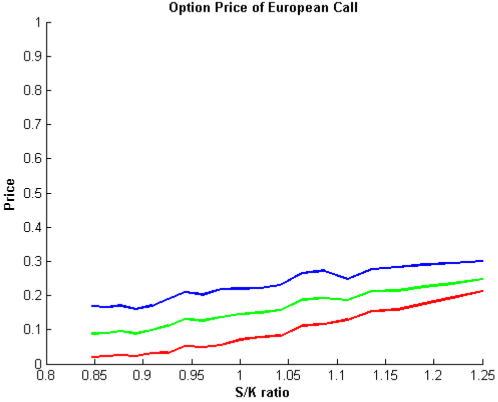


Figure: 2 $S_0 = 1, T = 1.0, \theta_1 = 10, \theta_2 = 10, p = 0.5,$ mean normal $=$ $0,$ sigma normal $= 0.1$

Numerical Simulations

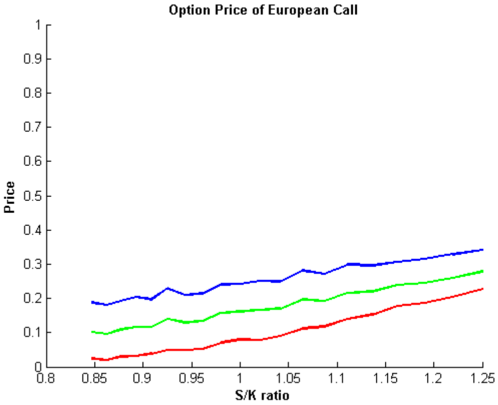


Figure: 3 $S_0 = 1, T = 1.2, \theta_1 = 10, \theta_2 = 10, p = 0.5,$ mean normal $=$ $0,$ sigma normal $= 0.1$

Numerical Simulations

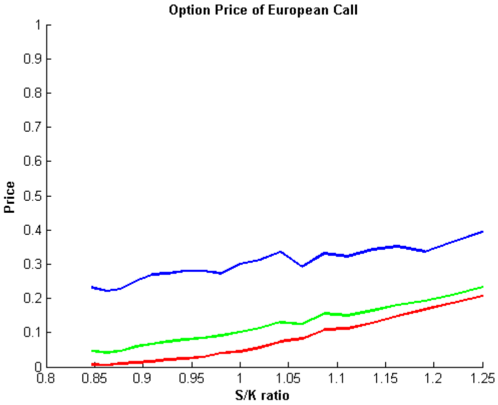


Figure: 4 $S_0 = 1, T = 0.5, \theta_1 = 5, \theta_2 = 10, p = 0.5,$ mean normal $=$ $0,$ sigma normal $= 0.1$

Numerical Simulations

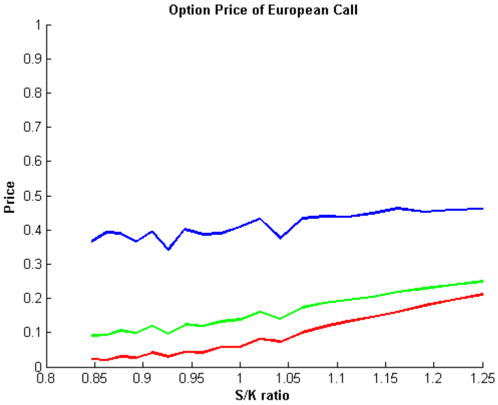


Figure: 5 $S_0 = 1, T = 1.0, \theta_1 = 5, \theta_2 = 10, p = 0.5,$ mean normal $=$ $0,$ sigma normal $= 0.1$

Numerical Simulations

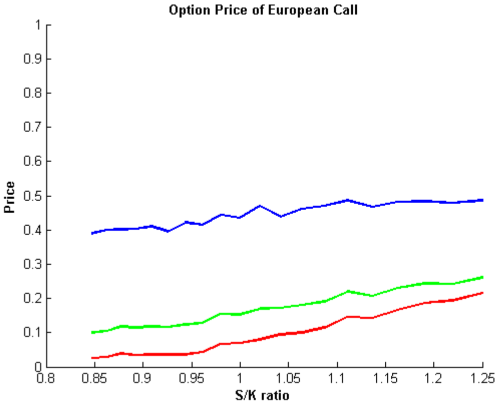
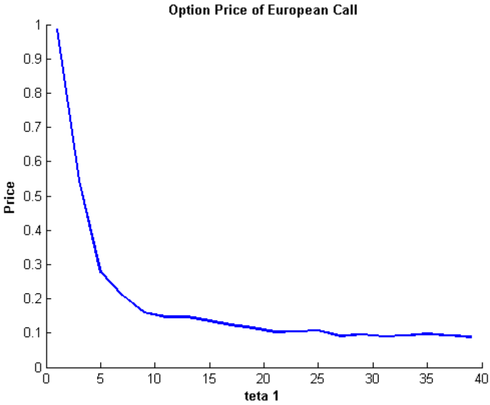


Figure: 6 $S_0 = 1, T = 1.2, \theta_1 = 5, \theta_2 = 10, p = 0.5,$ mean normal $=$ $0,$ sigma normal $= 0.1$

Numerical Simulations

If we fix the value of the θ_2 parameter in the double exponential distribution with $S/K = 1$ the corresponding plot is given in Figure 7.



Bo *et al.* (2010) Results

This research generalizes the results by L. Bo, Y. Wang, X. Yang 'Markov-modulated jump-diffusion for currency option pricing' published in *Insurance: Mathematics & Economics*, 2010. The jump process here was modeled as a compound Poisson process with log-normal amplitude to describe the jumps.

Conclusions

- We generalized the formulas of Bo *et al.* (2010) for a general Lévy process
- We applied obtained formulas to the case of the double exponential distribution of jump size
- We also provided numerical simulations of European call foreign exchange option prices for different parameters

Publications

1. Swishchuk, A.V., Tertychnyi M.V. and Elliott R.J. (2014): Pricing Currency Derivatives with Markov-modulated Lévy Dynamics. *Insurance: Mathematics and Economics, 2014*.
2. Also availabale on *arXiv: 1402.1953*.

Thank You for Your Attention and Time!

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