The Sum–Product Structure as a Mechanism for Risk Management

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A Sum–Product Structure

We are interested in the following sum-product structure:

$$L_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \qquad n \in \mathbb{N}. \tag{1}$$

- $\{X_i\}$: real-valued random variables
- $\{Y_j\}$: positive random variables

A Sum–Product Structure

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- $\{Y_j\}$: positive random variables

In this stochastic structure, $\{X_i\}$ and $\{Y_j\}$ interact in a transparent way, providing great convenience and flexibility to introduce dependence structures to the study.

We show how the sum–product structure (1) emerges naturally in risk management for insurance and finance.

Insurance and Financial Risks

When an insurance company invests its wealth in a financial market, it is exposed to two kinds of risks:

- insurance risk: the traditional liability risk, namely insurance claims, related to the insurance portfolio
- financial risk: the asset risk related to the investment portfolio including inflations of economy and stock market crashes

Both risks may impair the solvency of the insurance company.

We shall show examples from risk modeling, in which $\{X_i\}$ and $\{Y_j\}$ in the sum–product structure (1) represent the two kinds of risks.

Literature Review

- Norberg (1999, *SPA*)
- Paulsen (1993, *SPA*)
- Nyrhinen (1999, 2001, *SPA*)
- Kalashnikov and Norberg (2002, *SPA*)
- Frolova, Kabanov and Pergamenshchikov (2002, *FS*)
- T. and Tsitsiashvili (2003, *SPA*)
- Pergamenshchikov and Zeitouny (2006, *SPA*)
- Paulsen (2008, *PS*)
- Asmussen and Albrecher (2010, *Ruin probabilities*)
- Yang and Konstantinides (2014, *SAJ*)
- ⋮

Connection with Random Difference Equations

If (X, Y), (X_1, Y_1), (X_2, Y_2), . . . , are i.i.d., then

$$\begin{aligned} L_n &\overset{\mathrm{d}}{=} \sum_{i=1}^{n} X_i \prod_{j=i}^{n} Y_j \\ &= \left(\sum_{i=1}^{n-1} X_i \prod_{j=i}^{n-1} Y_j + X_n \right) Y_n \\ &\overset{\mathrm{d}}{=} (L_{n-1} + X_n)\, Y_n. \end{aligned}$$

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The (weak) limit L_∞, if it exists, satisfies

$$L_\infty \overset{\mathrm{d}}{=} \left(L_\infty + X \right) Y,$$

where L_∞ and (X, Y) on the right-hand side are independent.

- Kesten (1973, *AM*)
- Vervaat (1979, *Advances in AP*)
- Goldie (1991, *AAP*)
- Grey (1994, *AAP*)
- Goldie and Grübel (1996, *Advances in AP*)
- Collamore (2009, AAP)
- Enriquez, Sabot and Zindy (2009, *PTRF*)
- Li and T. (2014, *B*)
- ⋮

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Insurance without Considering Economic Factors

Consider a discrete-time risk model. Within period n:

- the total premium income is denoted by a non-negative random variable A_n
- the total claim amount plus other daily costs is denoted by another non-negative random variable B_n

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In the world without economic factors, the insurer's wealth process U_n exhibits a random walk structure:

$$U_0 = x > 0, \qquad U_n = U_{n-1} + (A_n - B_n).$$

An Investment Portfolio

Suppose that a financial market consists of a risk-free bond with price $S_{0,n}$ and d risky stocks with prices $S_{1,n}, \ldots, S_{d,n}$ at time n.

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Denote by $\pi_{0,n}$ the proportion invested in the bond and by $\pi_{k,n}$ the proportion invested in stock k. Thus, $\boldsymbol{\pi}_n = (\pi_{0,n}, \pi_{1,n}, \ldots, \pi_{d,n})$ is a stochastic process satisfying

$$\boldsymbol{\pi}_n{}^\top \mathbf{1} = \pi_{0,n} + \pi_{1,n} + \cdots + \pi_{d,n} = 1$$

and other possible constraints.

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and other possible constraints.

Denote by V_n the value process of this investment portfolio. It holds that

$$\frac{V_n - V_{n-1}}{V_{n-1}} = \sum_{k=0}^{d} \pi_{k,n} \frac{S_{k,n} - S_{k,n-1}}{S_{k,n-1}}, \tag{2}$$

or, equivalently,

$$\frac{V_n}{V_{n-1}} = \sum_{k=0}^{d} \pi_{k,n} \frac{S_{k,n}}{S_{k,n-1}}.$$

The Insurer's Wealth Process

The insurer's wealth process $\{U_n^{(\pi)}\}$, starting with $U_0^{(\pi)} = x > 0$, evolves according to

$$U_n^{(\pi)} = U_{n-1}^{(\pi)} \frac{V_n}{V_{n-1}} + (A_n - B_n) = U_{n-1}^{(\pi)} Y_n^{-1} - X_n, \tag{3}$$

- $X_n = B_n - A_n$: the net loss over period n, insurance risk
- $Y_n = V_{n-1}/V_n$: the overall stochastic discount factor over period n, financial risk

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Iterating (3) yields

$$U_n^{(\pi)} = \left(x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \right) \left(\prod_{j=1}^{n} Y_j^{-1} \right) = \left(x - L_n^{(\pi)} \right) \left(\prod_{j=1}^{n} Y_j^{-1} \right).$$

The quantity $L_n^{(\pi)}$ denotes the stochastic present value of the insurer's aggregate net losses. It exhibits a sum-product structure.

The Ruin Probability

The classical ruin probability by time $T \leq \infty$ is

$$\begin{aligned}\psi_\pi(x;T) &= P\left(\inf_{0\leq n\leq T} U_n^{(\pi)} < 0 \middle| x\right)\\ &= P\left(\inf_{0\leq n\leq T}\left(x - L_n^{(\pi)}\right)\left(\prod_{j=1}^{n} Y_j^{-1}\right) < 0 \middle| x\right)\\ &= P\left(\sup_{0\leq n\leq T} L_n^{(\pi)} > x\right),\end{aligned}$$

which is the tail probability of the maximum of $L_n^{(\pi)}$.

The Default Probability

The threshold 0 above does not make sense in practice. Insurance is a regulated business and it regulators will not allow an insurance company to continue its business if its wealth stays at a too low level.

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Letting a be a default threshold, the default probability by time $T \leq \infty$ is defined to be

$$\psi_{\pi}(x; a, T) = P\left(\inf_{0 \leq n \leq T} U_n^{(\pi)} < a \middle| x \right).$$

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For a certain value of a, it becomes the absolute ruin probability; see:

- Embrechts and Schmidli (1994, *Advances in AP*)
- Cai (2007, *Advances in AP*)
- Konstantinides, Ng and T. (2010, *JAP*)
- ⋮

The Default Probability (Cont'd)

Instead of (3) we use its equivalent form

$$U_n^{(\pi)} - a = \left(U_{n-1}^{(\pi)} - a \right) Y_n^{-1} - \left(X_n - aY_n^{-1} + a \right).$$

Write $\tilde{X}_n = X_n - aY_n^{-1} + a$. Iterating this recursive equation yields

$$U_n^{(\pi)} - a = \left(x - a - \sum_{i=1}^{n} \tilde{X}_i \prod_{j=1}^{i} Y_j \right) \prod_{j=1}^{n} Y_j^{-1} = \left(x - a - \tilde{L}_n^{(\pi)} \right) \prod_{j=1}^{n} Y_j^{-1}.$$

Note that $\tilde{L}_n^{(\pi)}$ appearing above also exhibits a sum-product structure.

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Note that $\tilde{L}_n^{(\pi)}$ appearing above also exhibits a sum-product structure.

Thus,

$$\psi_\pi(x; a, T) = P\left(\sup_{0 \le n \le T} \tilde{L}_n^{(\pi)} > x - a \right).$$

The dependence structure between $\tilde{X}_n$ and Y_n becomes complex.

Contents

Regularly Varying Distributions

For a distribution F on $\mathbb{R}$, we write $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$ if

$$\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x), \qquad y > 0.$$

The union

$$\mathcal{R} = \bigcup_{\alpha \geq 0} \mathcal{R}_{-\alpha}$$

forms one of the most important classes of heavy-tailed distributions.

- Bingham, Goldie and Teugels (1987, *Regular Variation*)
- Resnick (1987, *Extreme Values, Regular Variation, and Point Processes*)

Claims Follow AR(1)

- Denote the claim amount (plus other expenses) paid by an insurer within period i by a nonnegative random variable B_i
- Assume that these claim amounts form an AR(1) process: starting with a deterministic value $B_0 \geq 0$,

$$B_i = \rho B_{i-1} + \xi_i, \tag{4}$$

  - the autoregressive coefficient ρ takes value in $[0, 1)$
  - innovations $\{\xi_i\}$ are i.i.d. copies of a nonnegative random variable ξ

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An advantage of the AR(1) model is that it can capture asymptotic dependence between claim amounts:

Lemma

If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, then it holds for all $i_1, i_2 \in \mathbb{N}$ that

$$\lim_{x \to \infty} P\left(B_{i_2} > x \middle| B_{i_1} > x\right) = \rho^{|i_2 - i_1|\alpha}.$$

Log-return Rates also Follow AR(1)

- Suppose that there is a discrete-time financial market consisting of:
  - a risk-free bond with a deterministic continuously compounded rate of interest $r > 0$
  - a risky stock with a stochastic log-return rate $R_i \in \mathbb{R}$ during period i

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  - a risk-free bond with a deterministic continuously compounded rate of interest $r > 0$
  - a risky stock with a stochastic log-return rate $R_i \in \mathbb{R}$ during period i
- These log-return rates are also assumed to follow an AR(1) process: starting with a deterministic value R_0,

$$(R_i - \mu_R) = \gamma (R_{i-1} - \mu_R) + \eta_i, \tag{5}$$

  - the autoregressive coefficient γ takes value in $(-1, 1)$
  - the innovations $\{\eta_i\}$ are i.i.d. copies of a real-valued random variable η with mean 0
  - the constant μ_R is the mean of the stationary solution R_∞

A Bivariate AR(1) Risk Model

- Assume that $\{\xi_i\}$ and $\{\eta_i\}$ are mutually independent and so are the two AR(1) processes (4) and (5).
- Suppose that at the beginning of each period the insurer invests a fixed proportion $\pi \in [0,1]$ in the stock and keeps the rest in the bond. Then the wealth process $\{U_m^{(\pi)}\}$ evolves according to

$$U_m^{(\pi)} = \left((1-\pi)\mathrm{e}^r + \pi \mathrm{e}^{R_m}\right) U_{m-1}^{(\pi)} - (B_m - a),$$

  - $U_0^{(\pi)} = x > 0$: a deterministic initial value
  - $a > 0$: a constant premium amount during each period

Literature Review

- Cai (2002, *JAP*)
- Gerber (1981, *Bulletin of the Association of Swiss Actuaries*)
- Gerber (1982, *IME*)
- Mikosch and Samorodnitsky (2000, *AAP*)
- Wilkie (1986, *Transactions of the Faculty of Actuaries*)
- Wilkie (1987, IME)
- Yang and Zhang (2003, *PEIS*)

Our Main Result

Theorem (T. and Yuan (2012, NAAJ))

Assume that $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. Then it holds for every $n \in \mathbb{N}$ that

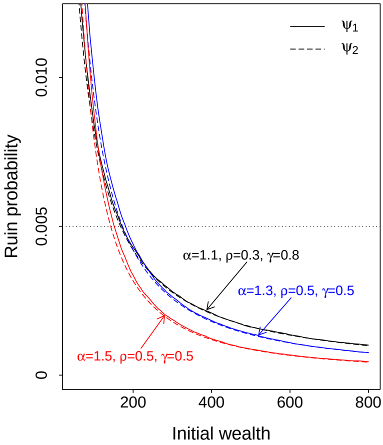
$$\psi(x;n) \sim E\left[\overline{F}\left(x + \sum_{i=1}^{n} \theta_i \left(a - \rho^i B_0\right)\right) \sum_{j=1}^{n} \left(\sum_{i=j}^{n} \theta_i \rho^{i-j}\right)^{\alpha}\right].$$

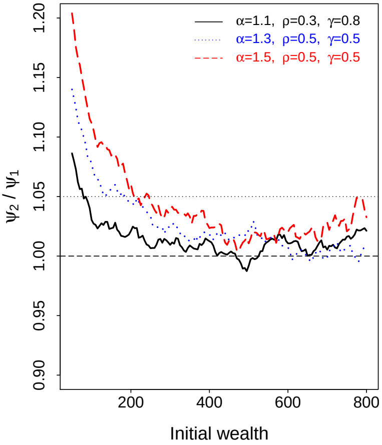
Modeling Specifications

- ξ in (4) follows Pareto with shape parameter $\alpha > 1$
- η in (5) follows normal with mean 0 and variance σ^2
- The parameters are set to
  - $\pi = 0.2$, 0.5 or 0.8
  - $n = 4$
  - $\alpha = 1.1$, 1.3 or 1.5
  - $\rho = 0.3$ or 0.5
  - $B_0 = 5.0$, $E[B] = 5.0$, $E[\xi] = (1 - \rho)E[B]$
  - $a = 5.5$
  - $r = 1.242\%$ (so that $\mathrm{e}^r - 1 = 1.25\%$)
  - $\gamma = 0.5$ or 0.8
  - $R_0 = 1.5\%$, $E[R] = 1.5\%$, $\sigma = 0.2$
- The sample size is $N = 1,000,000$ for both simulations and asymptotics despite the fact that it is more than enough for asymptotics.

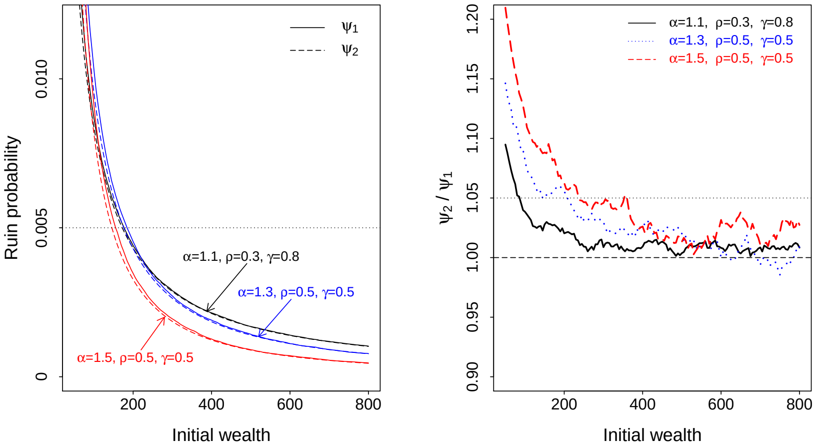
Accuracy of the Asymptotic Estimate

Graph 4.1(a) Accuracy of the asymptotic estimate ψ_2 for $\pi = 0.2$

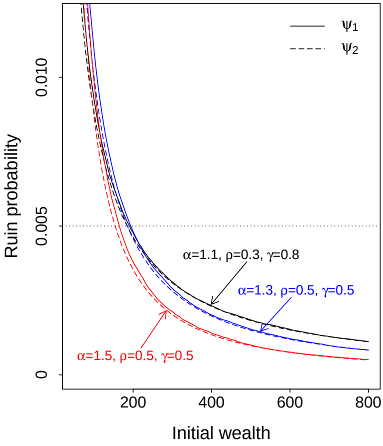


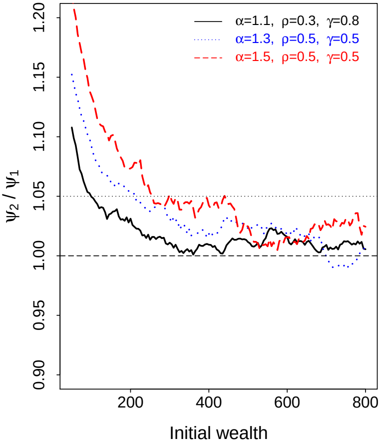


Graph 4.1(b) Accuracy of the asymptotic estimate ψ_2 for $\pi = 0.5$

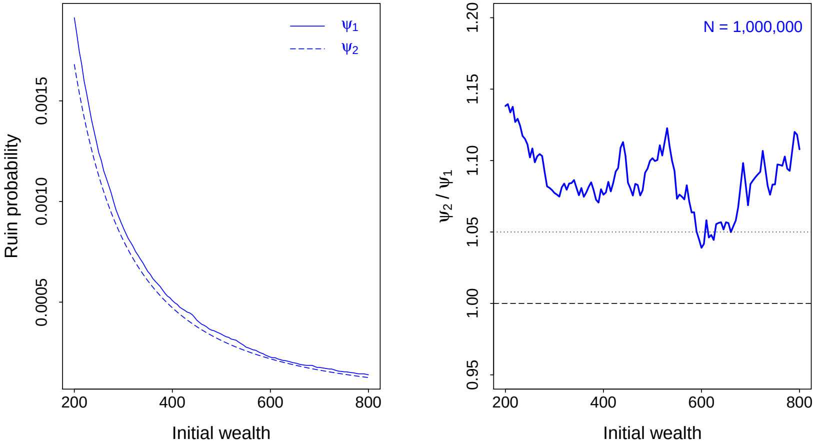


Graph 4.1(c) Accuracy of the asymptotic estimate ψ_2 for $\pi = 0.8$

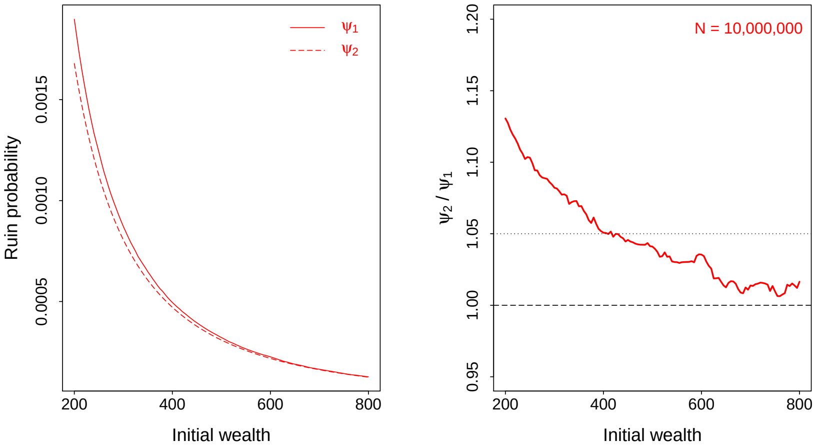




Graph 4.2(a) Accuracy of the asymptotic estimate ψ_2 for $\alpha = 2.0$ (N = 1,000,000)



Graph 4.2(b) Accuracy of the asymptotic estimate ψ_2 for $\alpha = 2.0$ (N = 10,000,000)



Portfolio Optimization

Generally speaking, investors aim at maximizing gains while minimizing risks. How to balance between gains and risks is a question that permeates many areas of finance.

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In the insurance context, due to the increasing prudence of insurance regulations, a solvency constraint needs to be imposed in portfolio optimization problems.

Our goal is to determine a value of π that maximizes the expected terminal wealth subject to a constraint on $\psi(x; n)$:

$$\begin{cases} \arg\max\limits_{0 \leq \pi \leq 1} E\left[U_n^{(\pi)}\right], \\ \text{subject to } \psi(x; n) \leq 1 - q, \end{cases}$$

where $0 < q < 1$ is chosen to be close to 1, say, $q = 0.995$.

Literature Review

- Paulsen (2003, *FS*)
- Irgens and Paulsen (2005, *SAJ*)
- Dickson and Drekic (2006, *AAS*)
- Kostadinova (2007, IME)
- He, Hou, and Liang (2008, *IME*)

Modeling Specifications

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- η in (5) follows normal with mean 0 and variance σ^2
- The parameters are set to
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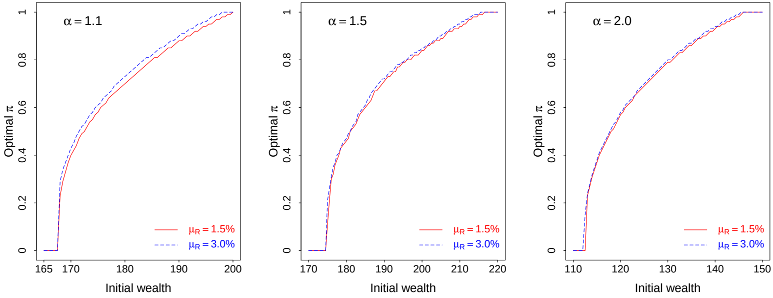
  - $\rho = 0.3$
  - $B_0 = 5.0$, $E[B] = 5.0$, $E[\xi] = (1 - \rho)E[B]$
  - $a = 5.5$
  - $r = 1.242\%$
  - $\gamma = 0.8$,
  - $R_0 = 1.5\%$, $E[R] = 1.5\%$ or 3%
  - $\sigma = 0.2$.

Numerical Results

Graph 5.1 The optimal π for different values of the initial wealth



Thank You Very Much!!!