Value-at-Risk Bounds with Variance Constraints

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Motivation

- A portfolio $(X_1, X_2, ..., X_n)$:

Full information on **marginal distributions**:
$X_j \sim F_j$ and represent risks as $X_j = F_j^{-1}(U_j)$.

$+$

Full Information on **dependence**:
$(U_1, U_2, ..., U_n) \sim C$ (C is called the copula)

$=$

$\text{VaR}_q [X_1 + X_2 + ... + X_n]$ can be computed!

- A portfolio $(X_1, X_2, ..., X_n)$:

Full information on **marginal distributions**:
$X_j \sim F_j$ and represent risks as $X_j = F_j^{-1}(U_j)$.

$+$

Partial Information on **dependence**:
$(U_1, U_2, ..., U_n) \sim$?

$=$

$\text{VaR}_q [X_1 + X_2 + ... + X_n]$ can**not** be computed!

Literature

• Makarov (1981), Rüschendorf (1982), Rüschendorf & Uckelmann (1991), Denuit, Genest & Marceau (1999), Embrechts & Puccetti (2006) Embrechts, Puccetti & Rüschendorf (2013):

$$M := \mathsf{Sup}\left\{\mathsf{VaR}_q\left[X_1 + X_2 + ... + X_n\right]\right\},\\ \text{subject to}\quad X_j \sim F_j.$$

• **Explicit sharp (attainable) bounds**

· $n = 2$ (Makarov, Rüschendorf)

· homogeneous portfolios under some conditions

· Asymptotic sharpness results

• **Approximate sharp bounds**

· The Rearrangement Algorithm (Puccetti & Rüschendorf)

Example of "M"

• Consider a portfolio of 10,000 loans all having a default probability $p = 0.049$. The default correlation is $\rho = 0.0157$. We plot VaR_q when using the KMV credit risk portfolio model (Industry standard - also used in Basel III and Solvency II) and we compare it with M.

confidence	VaR_q	"M"
$q = 0.95$	10.1%	98%
$q = 0.995$	15.1%	100%

Some observations

• One has that:

$$M \geq \text{VaR}_q[X_1] + \text{VaR}_q[X_1] + ... + \text{VaR}_q[X_n]$$

(RHS=situation of perfect dependence, i.e. when all $U_i = U$)

• So, the **worst case VaR (i.e.** M**) corresponds to a portfolio in which diversification does not pay off**.

Dependence

- Consider the problem:

$$\boxed{\begin{array}{l} M := \mathsf{sup}\left\{\mathsf{VaR}_q\left[X_1 + X_2 + ... + X_n\right]\right\}, \\ \text{subject to } \; X_j \sim F_j, \mathsf{var}(X_1 + X_2 + ... + X_n) \leq s^2 \end{array}}$$

Results

• Getting **simple to compute upper (and lower) VaR bounds**.

• Getting a very **practical algorithm** that enables the practical computation of (approximate) sharp VaR bounds.

• Showing that the approximate VaR bounds are typically close to the simple theoretical bounds.

• Showing that in the presence of a constraint on the variance, the VaR bounds can **significantly improve** upon the unconstrained bounds.

• Establishing a **connection** between **VaR bounds** and **convex lower bounds**.

The Unconstrained Case ($s^2 = \infty$)

Upper bound for VaR with given marginals

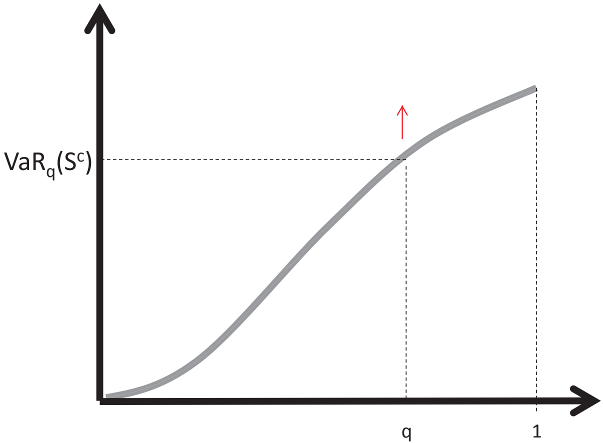
$$\mathrm{VaR}_q\left[X_1 + X_2 + ... + X_n\right] \leqslant B := \mathrm{TVaR}_q\left[X_1^c + X_2^c + ... + X_n^c\right]$$

where

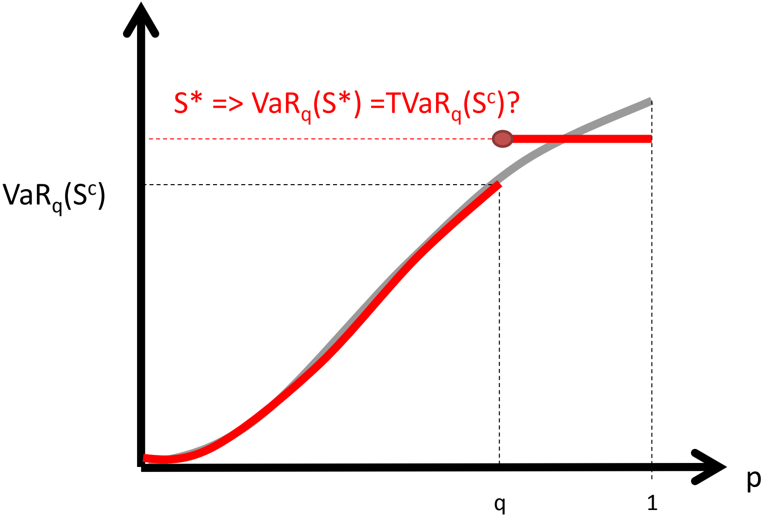
$$\mathrm{TVaR}_q\left[X\right] = \frac{1}{1-q}\int_q^1 VaR_p\left[X\right]\ \mathrm{d}p,$$

Proof:

$$\begin{aligned}\mathrm{VaR}_q\left[X_1 + X_2 + ... + X_n\right] &\leqslant \mathrm{TVaR}_q\left[X_1 + X_2 + ... + X_n\right] \\ &\leqslant \mathrm{TVaR}_q\left[X_1^c + X_2^c + ... + X_n^c\right]\end{aligned}$$



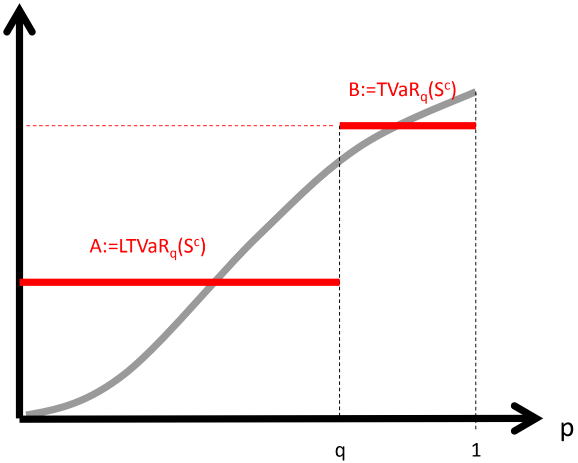
$VaR_q(S^c)$
q
1



S* => VaRq(S*) =TVaRq(Sc)?
VaRq(Sc)
p
q
1

Unconstrained Bounds with $X_j \sim F_j$

$$A = LTVaR_q(S^c) \leqslant \mathrm{VaR}_q\left[X_1 + X_2 + \ldots + X_n\right] \leqslant B = TVaR_q(S^c)$$



The Rearrangement Algorithm (RA)

• The rearrangement algorithm (RA) (Puccetti & Rüschendorf, 2012) can be seen as a very clever attempt to obtain "sums that behave as much as possible as sums "that are flat in the upper tail". It can be used as a practical (approximative) way to obtain the true upper bound for the VaR.

• Let d be the number of points used to discretize the risks with distribution F_j. ($j = 1, 2, ..., n$). One first samples the risks into d equally probable values $x_{\mathbf{ij}}$ and one obtains a $d \times n$ matrix $\mathbf{X} = (x_{\mathbf{ij}})$.

• Loosely speaking, the RA is then a method in which subsequent columns of the appropriate lower matrix are rearranged such that they become (locally) anti-monotonic with the sum of all other columns until convergence is reached.

q				
1-q	8	0	3	Sum= 11
	10	1	4	Sum= 15
	11	7	7	Sum= 25
	12	8	9	Sum= 29

q

1-q

8	0	3	Sum= 11
10	1	4	Sum= 15
11	7	7	Sum= 25
12	8	9	Sum= 29

Rearrange **within** columns..to make the sums as constant as possible…

B=(11+15+25+29)/4=20

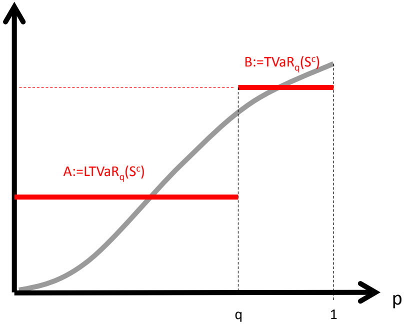
q				
1-q	**8**	**8**	**4**	Sum= 20
	10	**7**	**3**	Sum= 20
	12	**1**	**7**	Sum= 20
	11	**0**	**9**	Sum= 20

=B!

VaR bound with variance constraint ($s^2 < \infty$)

- Define a random variable X^* as follows:

$$X^* = \begin{cases} A & \text{with probability } q \\ B & \text{with probability } 1-q. \end{cases} \tag{2}$$

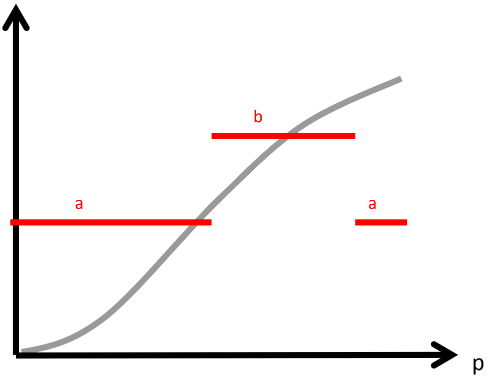


- If $\mathrm{var}(X^*) \leqslant s^2$, then the bound B cannot be readily improved. When $\mathrm{var}(X^*) > s^2$, B is too wide. $\Rightarrow$ The idea is as follows:

Solving with the variance constraint

- Define a random variable Y^* as follows:

$$Y^* = \begin{cases} a & \text{with probability } q \\ b & \text{with probability } 1-q \end{cases},$$



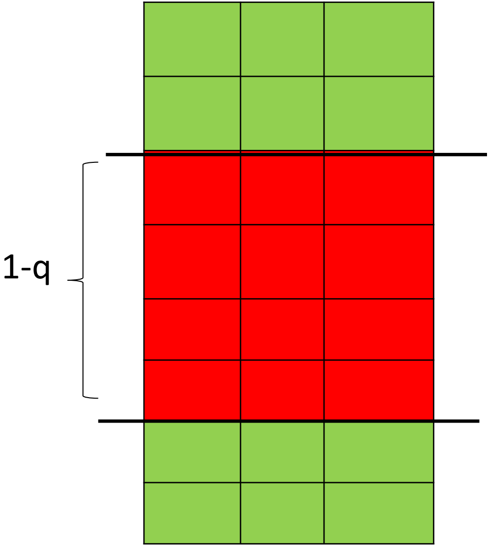
Constrained Bounds with $X_j \sim F_j$ and variance $\leqslant s^2$

$$a = \max\left(A, \mu - s\sqrt{\frac{1-q}{q}}\right) \leqslant \mathrm{VaR}_q\left[X_1 + X_2 + ... + X_n\right]$$

$$\leqslant b = \min\left(B,\ \mu + s\sqrt{\frac{q}{1-q}}\right)$$

- Hence, if the variance s^2 is not "too large" (i.e. when $s^2 \leqslant q(A-\mu)^2 + (1-q)(B-\mu)^2$), then the bound b strictly improves upon B.

Extended RA (ERA)



-Apply the RA separately on the appropriate red area (with average of the sums =b) and the remaining red green area

-If the variance constraint is satisfied then stop the algorithm, otherwise shift up the red area by one row and start over.

Bounds on VaR of sum of Pareto ($\theta = 3$) with $\rho = 0.15$

Panel A: Approximate sharp bounds obtained by the ERA

(m_d, M_d)		$n = 10$	$n = 100$
$\mathrm{VaR}_{95\%}$	$d = 1,000$	(4.118 ; 19.93)	(42.55 ; 174.0)
$\mathrm{VaR}_{99.5\%}$	$d = 1,000$	(4.868 ; 53.99)	(47.07 ; 457.6)

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<u>Panel B:</u> Variance-constrained VaR bounds (theoretical)

(m_d, M_d)		$n = 10$	$n = 100$
$\mathrm{VaR}_{95\%}$,	$d = 1,000$	(4.100 ; 20.35)	(42.45 ; 175.9)
$\mathrm{VaR}_{99.5\%}$,	$d = 1,000$	(4.662 ; 54.87)	(47.06 ; 459.4)

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Panel C: Unconstrained VaR bounds (theoretical)

(m_d, M_d)		$n = 10$	$n = 100$
$\mathrm{VaR}_{95\%}$,	$d = 1,000$	(3.642 ; 29.05)	(36.42 ; 290.5)
$\mathrm{VaR}_{99.5\%}$,	$d = 1,000$	(4.615 ; 64.06)	(46.15 ; 640.6)

	(A_d, B_d)	(a_d, b_d)	(m_d, M_d)	KMV	Beta	CreditMetrics
$\text{VaR}_{0.8}$	(0%; 24.50%)	(3.54%; 10.33%)	(3.63%; 10%)	6.84%	6.95%	6.71%
$\text{VaR}_{0.9}$	(0%; 49.00%)	(4.00%; 13.04%)	(4.00%; 13%)	8.51%	8.54%	8.41%
$\text{VaR}_{0.95}$	(0%; 98.00%)	(4.28%; 16.73%)	(4.32%; 16%)	10.10%	10.01%	10.11%
$\text{VaR}_{0.995}$	(4.42%; 100.00%)	(4.71%; 43.18%)	(4.73%; 40%)	15.15%	14.34%	15.87%

Table 5.4 The table provides VaR bounds and VaR computed in different models (KMV, Beta, CreditMetrics).

Conclusions

- Assess uncertainty on Value-at-Risk of a portfolio with given marginals with partial information on dependence (through the variance of the sum)

- Other information on dependence
  - VaR bounds with higher moments (with Bernard, Denuit)

$$M := \sup \mathrm{VaR}_q \left[X_1 + X_2 + ... + X_n\right],$$
$$E((X_1 + X_2 + ... + X_n)^k) \text{ is known, } k = 1, 2, ..., m$$

  - Information on the joint distribution under some scenarios (with Bernard)

References

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