Levy Subordinated Hierarchical Archimedean Copula : Theory and Application

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(with Ken Seng Tan and Wenjun Zhu)

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- Elliptical dependence structures (Gaussian and Student's t Copulas) are widely used.

  => do not allow for correlation asymmetries

- Archimedean Copulas (AC) such as Gumbel and Calyton Copula are used to capture the asymmetric tail dependence.

Archimedean copulas

$$C(u_1, u_2, \ldots, u_d) = \psi[\psi^{-1}(u_1)+, \ldots, \psi^{-1}(u_d)],$$

is a d-dimensional Archimedean copula iff $\psi \in \mathcal{G}$ defined as

$$\left\{\psi : [0,\infty) \to [0,1] \,|\, \psi(0) = 1, \psi(\infty) = 0, (-1)^k \frac{d^k}{du^k}\psi(u) \geq 0, k \in \mathbb{N}\right\}$$

a set of *completely monotonic* (c.m.) functions.

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Drawbacks of AC

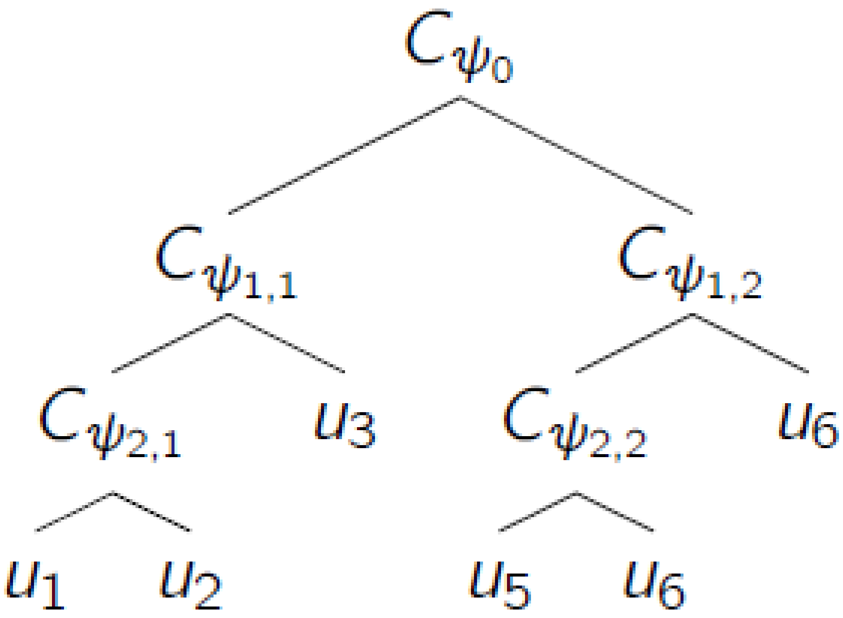
Exchangeable

- invariant under permutation

=>inappropriate in the multivariate model.

- Hierarchical Archimedean copula (HAC) has been proposed to overcome the disadvantage of Archimedean Copulas (AC).

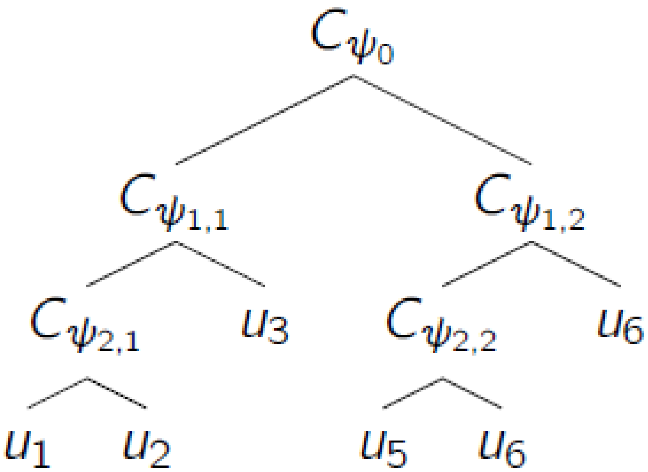
3-level, six-dimensional HAC



"Copula of Copulas" Structure

$$C^{(0)}_{0,1}(u_1, \cdots, u_6)$$

$$= C^{(0)}_{0,1}(C^{(1)}_{1,1}(C^{(2)}_{1,1}(u_1, u_2), u_3), C^{(1)}_{1,2}(C^{(2)}_{2,1}(u_4, u_5), u_6))$$



$$\psi_0, \psi_{1,1}, \psi_{1,2}, \psi_{2,1}, \psi_{2,2} \in \mathcal{G}, \tag{3}$$

$$(\psi_0^{-1} \circ \psi_{1,j})' \quad \text{and} \quad (\psi_{1,i}^{-1} \circ \psi_{2,j})' \in \mathcal{G}, \tag{4}$$

$$i = 1, 2, \quad j = 1, 2 \tag{5}$$

- Condition (4) is called **compatible condition**, adding difficulties in constructing HAC.
- Gumbel HAC satisfies Equation (4) and are easy to simulate. As a result, it is the most frequently used HAC in empirical study.

Levy Subordinated HAC

- Hering et al. (2010) first construct a two-level hierarchical model based on Lévy subordinators.

  Theorem 2.1:

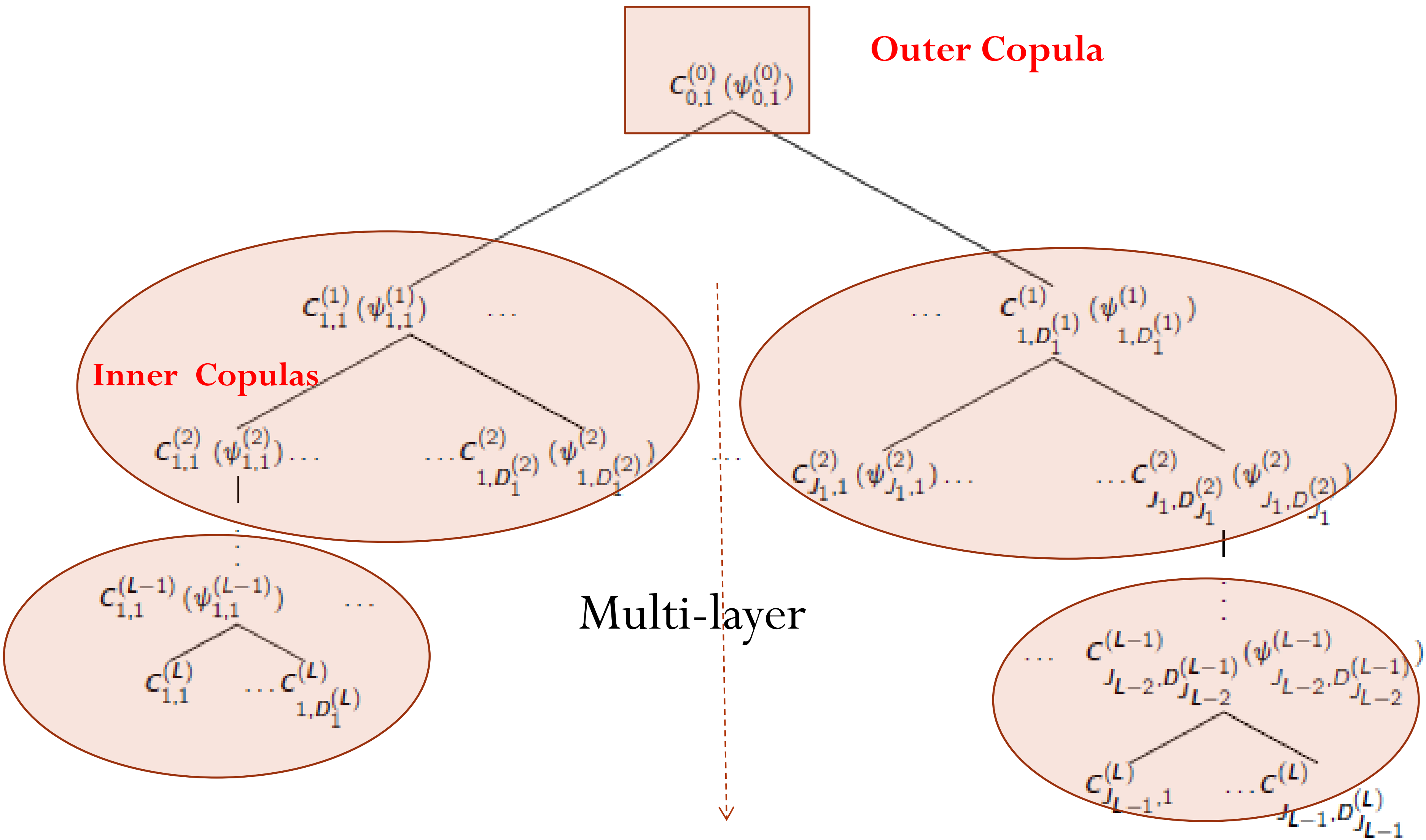
$$(\psi_0^{-1} \circ \psi_1)' \text{ is c.m.} \Leftrightarrow \psi_0^{-1} \circ \psi_1 = \Psi \text{ is the Laplace exponent of a Lévy subordinator.}$$

- Mai and Scherer (2012), introduced h-extendible copulas including a three-level LS-HAC case.

Main Goals

- Introduce a general framework of LS-HAC models with **arbitrary levels**.
- Propose a **three-stage** estimation procedure. We use hierarchical clustering analysis to determine LSHAC structure.
- **Empirically** examine performance of LS-HAC models.

Figure : General Framework of LS-HAC Model



Integral Representation => **For Simulation Purpose**

Theorem

Given the general structure of LS-HAC, the copula function can be constructed as the following:

$$\int_0^\infty \prod_{j_1=1}^{D_{s_0}^{(1)}} \int_0^\infty \prod_{j_2=1}^{D_{s_1}^{(2)}} \int_0^\infty \cdots \int_0^\infty \prod_{j_{L-1}=1}^{D_{s_{L-2}}^{(L-1)}} \int_0^\infty \prod_{j_L=1}^{D_{s_{L-1}}^{(L)}} \left(F_{s_{L-2} j_{L-1}}^{(L-1)} (\bar{u}_{s_{L-1} j_L}) \right)^{v_{s_{L-2} j_{L-1}}^{(L-1)}} (dG)_{j_{L-1}}^{(L-1)},$$

where we define $(l = 1, \ldots, L-1)$:

$$(dG)_{j_0}^{(0)} = dG_{0,1}^{(0)}(v_{0,1}^{(0)}),$$

and

$$(dG)_{j_l}^{(l)} = d\tilde{G}_{s_{l-1} j_l}^{(l)}(v_{s_{l-1} j_l}^{(l)}; v_{s_{l-2} j_{l-1}}^{(l-1)}) \ldots d\tilde{G}_{s_0 j_1}^{(l)}(v_{s_0 j_1}^{(1)}; v_{0,1}^{(0)}) dG_{0,1}^{(0)}(v_{0,1}^{(0)}).$$

Generators of LS-HAC

Proposition => **For Calibration Purpose**

For $1 \leq l \leq L$, in level l, the j_l-th copula generator in position s_{l-1}: $\psi^{(l)}_{s_{l-1} j_l}$, can be expressed as:

Outer generator

Lévy subordinators

$$\psi^{(l)}_{s_{l-1} j_l} = \psi^{(0)}_{0,1} \bigodot_{i=1}^{l-1} \tilde{\Psi}^{(i+1)}_{s_i j_{i+1}}, \quad (6)$$

Inner generator

where $\bigodot_{i=1}^{n} f_i := f_1 \circ \ldots \circ f_n$, and $\psi^{(l)}_{s_{l-1} j_l}$ is c.m..

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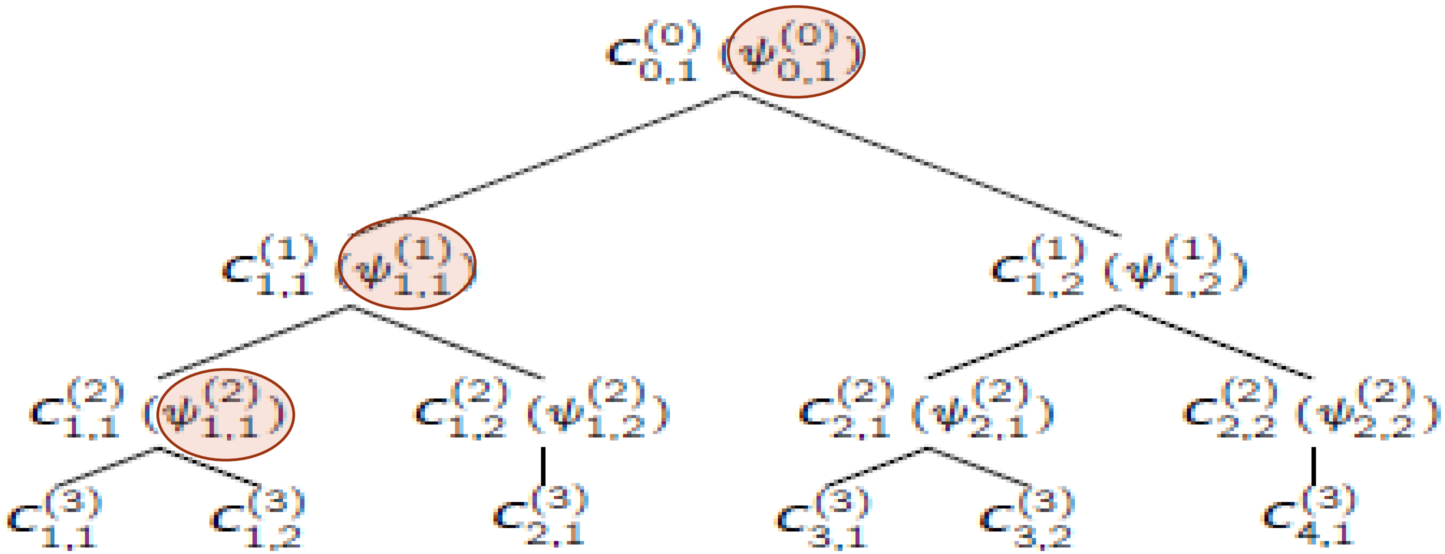
A three-layer Six-Dimensional LS-HAC



$$\begin{aligned} \psi_{1,1}^{(1)} &= \psi_{0,1}^{(0)} \circ \widetilde{\Psi}_{1,1}^{(1)}, \\ \psi_{1,1}^{(2)} &= \psi_{1,1}^{(1)} \circ \widetilde{\Psi}_{1,1}^{(2)} = \psi_{0,1}^{(0)} \circ \widetilde{\Psi}_{1,1}^{(1)} \circ \widetilde{\Psi}_{1,1}^{(2)} \end{aligned}$$

In the previous expressions, $\psi_{i,j}^{(l)}$ is AC generators and $\widetilde{\Psi}_{s,t}^{(l)}$ is Laplace exponent of Lévy subordinators.

LS-HAC

In this paper, we consider

- AC generators
  - Clayton (CL)
  - Gumbel (GM)
  - Inverse Gaussian (IG)

Name	$\psi(u)$
CL	$(1+u)^{-\frac{1}{\theta}}, \theta \geq 0$
GM	$\exp(-x^{\frac{1}{\theta}})$
IG	$\exp(\frac{1}{\theta}(1-\sqrt{1+2\theta^2 x}))$

LS-HAC

- Levy subordinators
  - Gamma (G)
  - Gumbel or Stable (GM)
  - Inverse Gaussian (IG)

Distribution	Inverse Gaussian (IG)	Gamma (G)
Characteristic Function (c.f.)	$\phi_{IG}(u) = \exp\left(-\omega\left(a\sqrt{2ui+b^2}-ab\right)\right)$	$\phi_G = \exp\left(-\omega a \log\left(1-iu/b\right)\right)$
Laplace Exponent	$\Psi_{IG}(u) = a\sqrt{2u+b^2}-ab$	$\Psi_G(u) = a\log(1+u/b)$

- ***Note that Gumbel HAC is a special case of LS-HAC when outer generator and subordinator are Gumbel.***

LS-HAC Estimation

- **Three-Stage Estimation**
  - Stage 1: Estimating Margins
  - **Stage 2: Determining Hierarchical Structure => Hierarchical Clustering Analysis**
  - Stage 3: Estimating LS-HAC parameters by using global MLE, instead of using sequential MLE in classical Gumbel HAC.

Stage 1: ARMA-GARCH-GH models

- Autoregressive moving average-generalized autoregressive conditional heteroskedasticity (ARMA-GARCH) models (Bollerslev, 1986; Engle, 1982) are used to analyze the dynamics of the margins for the financial data.

ARMA (m,n)-GARCH (p,q) Model

$$\begin{aligned} r_{i,t} &= c_i + \sum_{j=1}^{m} \phi_{i,j} r_{i,t-j} + \sum_{j=1}^{n} \theta_{i,j} \epsilon_{i,t-j} + \epsilon_{i,t}, \\ \epsilon_{i,t} &= \sqrt{h_{i,t}} z_{i,t}, \\ h_{i,t} &= \omega_i + \sum_{j=1}^{p} a_{i,j} h_{i,t-j} + \sum_{j=1}^{q} b_{i,j} \epsilon^2_{i,t-j}. \end{aligned}$$

Generalized Hyperbolic (GH)

- Barndorff-Nielsen (1977) uses the GH distribution for financial time series exhibiting skewness, leptokurtosis and tail-thickness.

- The GH distribution nests many well-known highly flexible distributions (normal, Student-*t*, hyperbolic ,Variance Gamma (VG),Normal Inverse Gaussian and GH skewed *t* distributions as special case or limit cases.

- After obtaining the marginal parameters, we use probability transform based on the best goodness-of-fit residual distribution to obtain the inputs of copulas.

$$u_{it} = F_{Z_i}(z_{it})$$

Stage 2: Hierarchical Clustering Analysis

- Despite its importance, few papers discuss the grouping method of the HAC or LSHAC models.

- Okhrin et. al. (2013) introduce an estimation procedure to determine optimal structure of HAC by evaluating all possible structures.
  - Time-consuming and inefficient to enumerate all possible structures, especially for high-dimensional problems.

- We employ hierarchical clustering method to determine the structure of LSHAC.

Dissimilarity (Proximity) Matrix

Using six dimension data as an example:

$$
\begin{array}{c}
\begin{matrix} \quad x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \\
\begin{pmatrix}
d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \\
d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & d_{2,6} \\
d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} & d_{3,6} \\
d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & d_{4,6} \\
d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & d_{5,6} \\
d_{6,1} & d_{6,2} & d_{6,3} & d_{6,4} & d_{6,5} & d_{6,6}
\end{pmatrix}
\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix}
\end{array}
$$

- $d_{i,j}$ represents dissimilarity of i and j .
- $d_{i,i}=0$ and $d_{i,j} = d_{j,i}$ (symmetric)
- High $d_{i,j}$ represents high degree of dissimilarity

Euclidean V.S. Dependence Distance:

- Euclidean Distance: $d_{i,j}^{\text{Euclid}} = \sqrt{\sum_k (x_{k,i} - x_{k,j})^2}$
- Dependence Distance $d_{i,j}^{\text{Corr}} = 1 - \mathfrak{R}_{i,j}$

where $\mathfrak{R}_{i,j}$ is an **association measure**, such as *Spearman's rho* and *Kendall's tau*.

- In literature, a sequential procedure according to Kendall's tau is used to determine hierarchical structure of classical Gumbel HAC.
- It ignores ***dissimilarity*** from "Euclidean Distance" .

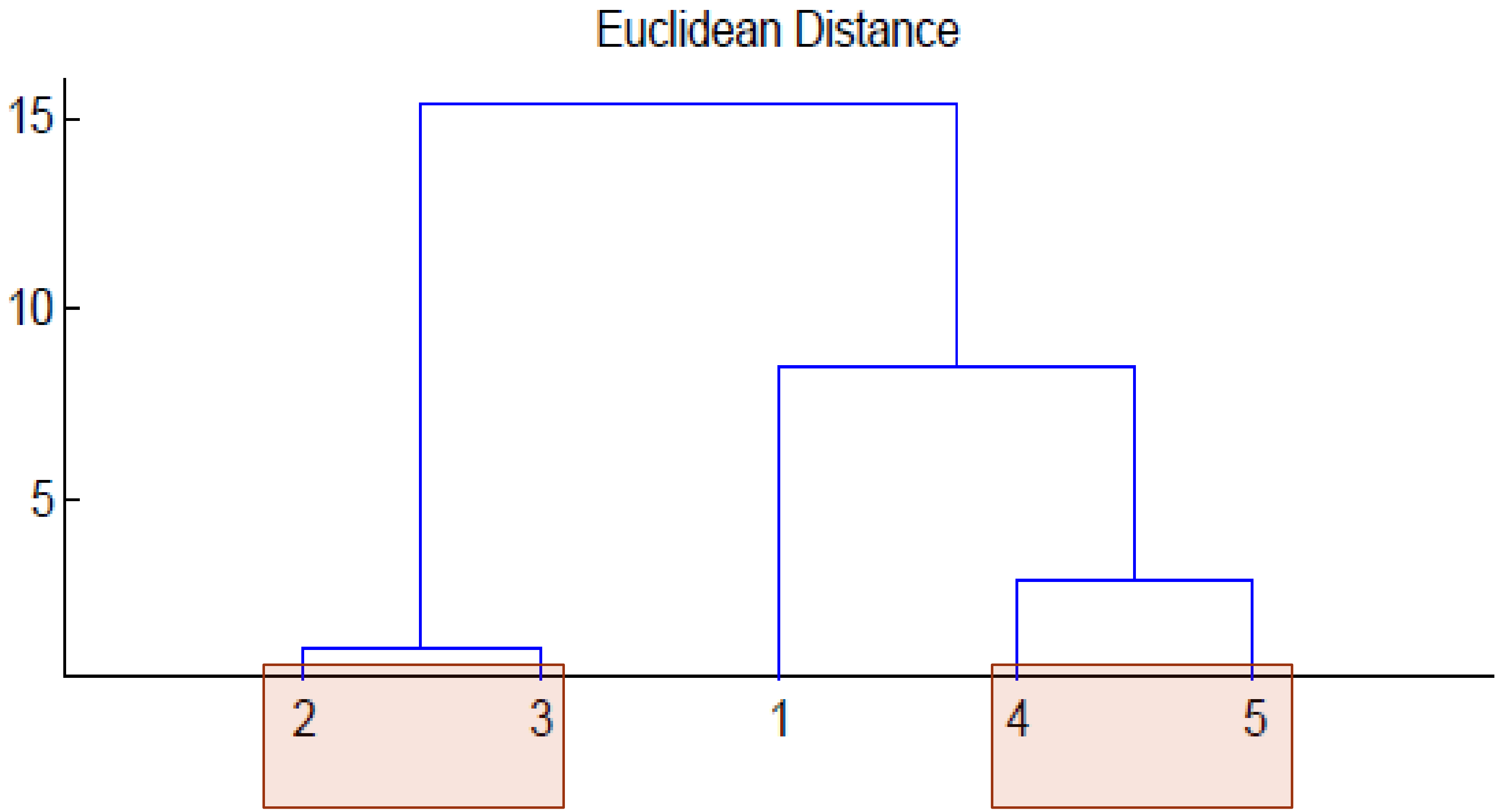
Counter Example

- This example shows the problem of the two distances.

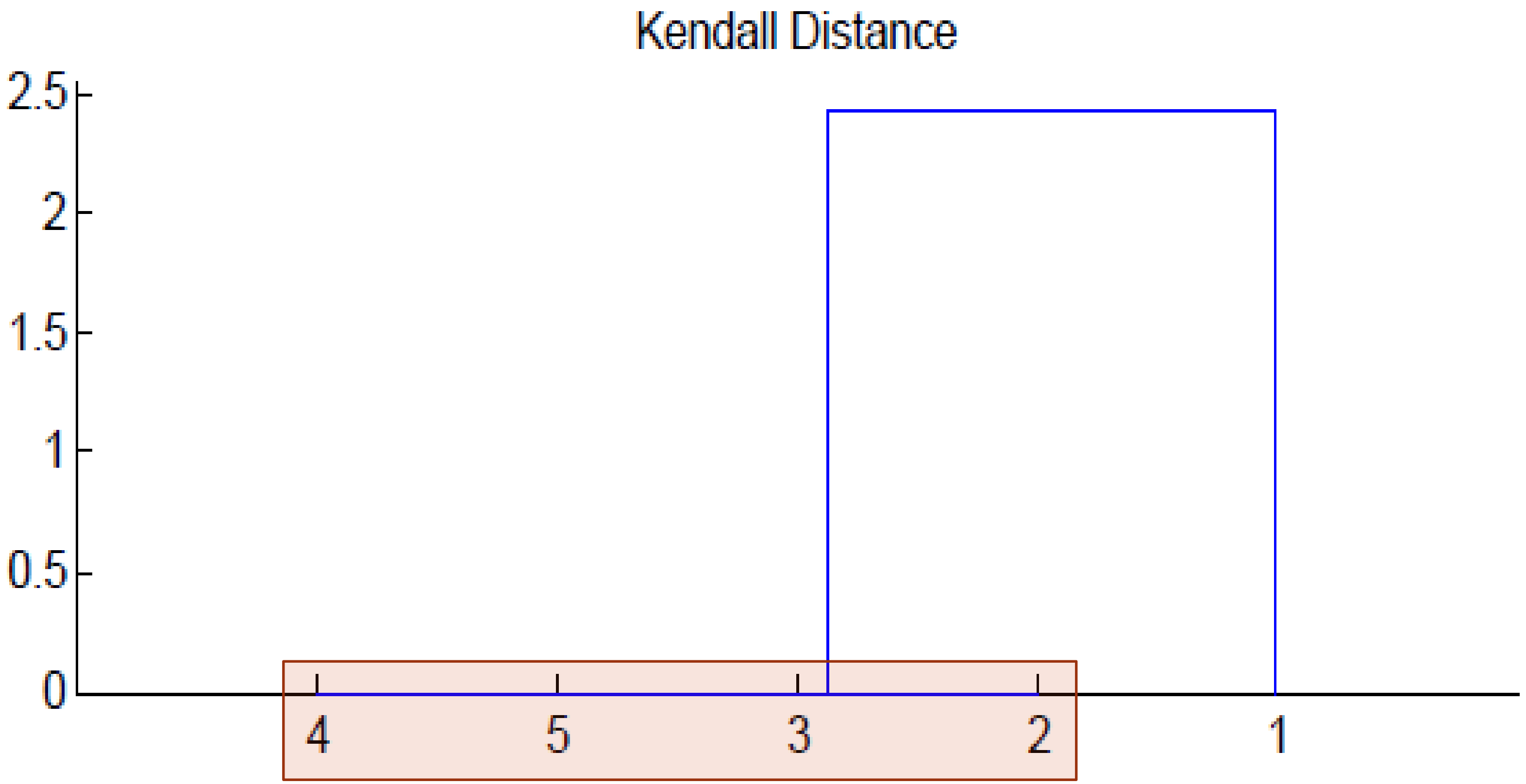
$$
\begin{aligned}
x_1 &= \text{rand}(1, 100), \\
x_2 &= 0.051 : 0.001 : 0.15, \\
x_3 &= 0.001 : 0.001 : 0.10, \\
x_4 &= 0.851 : 0.001 : 0.95, \\
x_5 &= 0.9 * (0.801 : 0.001 : 0.90)
\end{aligned}
$$

- Intuitively, we have
  - x2 and x3 => perfect positive correlated with small values
  - x4 and x5 => perfect positive correlated with large values
  - x1 => independent of others (third group)

Euclidean Distance



Kendall's Tau measure

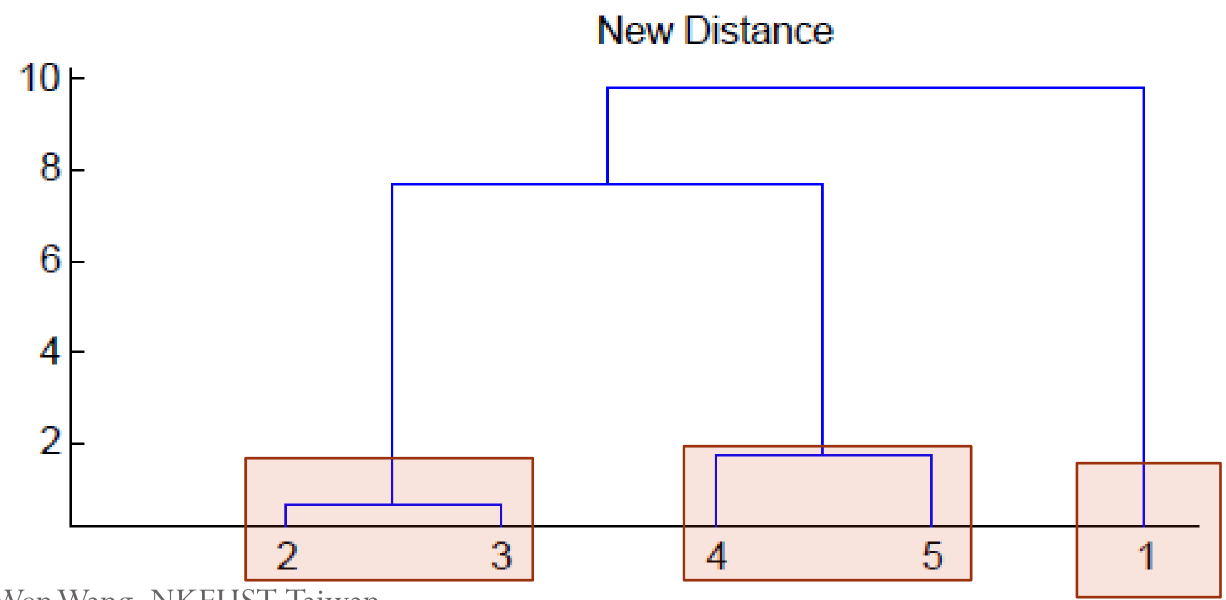


It groups *x2* to *x5* as a *perfect-positive-correlated* group without further discrimination

A Trade-off measure

- In this paper, we propose a new measure by dividing Euclidean distance by dependence distances

$$\frac{d_{i,j}^{\text{Euclid}}}{1+\tau_{i,j}}$$



Empirical Analysis

- Data: 5-year daily data of Stock indices
  - European (France, UK and Germany)
  - American (Brazil, Canada and US)

Stage 1: Margins

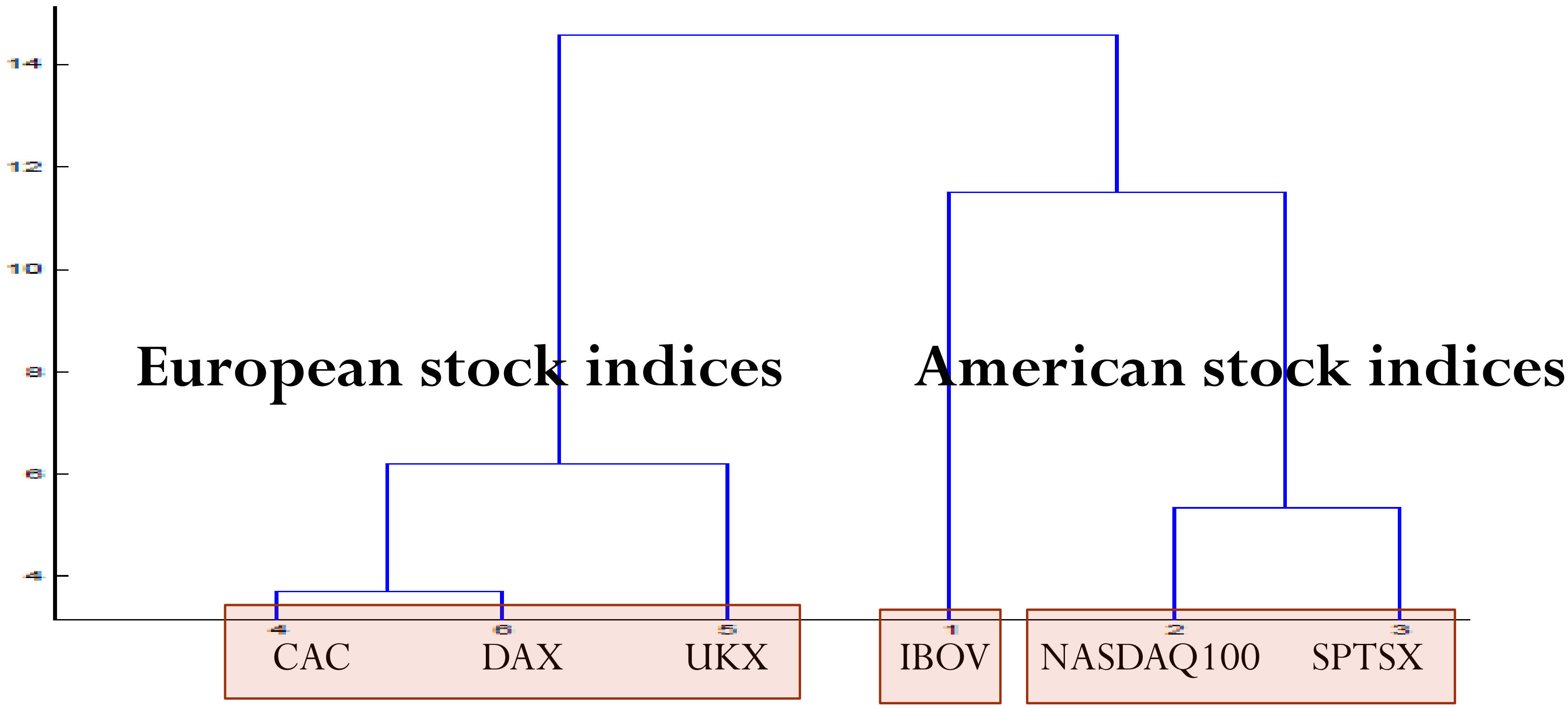
Marginal Estimation Results (GARCH(1,1) effect but no ARMA effect)

Indices	IBOV	NDX	SPTSX	CAC	UKX	DAX
Best Distri-bution	t	VG	VG	t	VG	VG
$c(\times 10^{-4})$	1.7676 (0.3753)	10.4844 (3.8384)***	2.4634 (0.8901)	9.5697 (2.4141)***	4.5534 (1.5001)	8.8959 (2.2612)**
$\omega(\times 10^{-6})$	5.617 (2.227)**	2.9194 (2.7788)***	3.9638 (0.9451)	2.7935 (1.526)*	8.2843 (1.2079)	1.3621 (1.1318)
a_1	0.9191 (53.6884)***	0.8775 (36.0184)***	0.9302 (62.5136)***	0.9254 (46.6514)***	0.9374 (51.7158)***	0.9363 (58.6861)***
b_1	0.0685 (4.3703)***	0.1048 (4.5191)***	0.0698 (4.3598)***	0.0683 (3.5849)***	0.0586 (3.3877)***	0.0597 (3.9366)***
v	8 (6.7306)***			6.5103 (3.5849)***		
α		1.9846 (11.8214)***	2.4289 (7.3098)***		2.4826 (7.5641)***	2.174 (9.3175)***
β		-0.1753 (11.8214)***	-0.415 (-3.3496)***		-0.1739 (-1.9242)***	-0.1739 (-1.9242)***

Empirical Analysis: Hierarchical clustering

Dendrogram according to our distance measure

Geographic Groups



Empirical Analysis: Global MLE

Table : Six Dimension Estimating Results

	Copula	Log-Likelihood	AIC	BIC
Elliptical	Gaussian	3361.4681	-3346.4681	-3309.3370
	TV-Gaussian	3384.0908	-3367.0908	-3325.0089
	T	3558.7891	-3542.7891	-3503.1826
	TV-T	3573.9428	-3555.9428	-3511.3855
LS-HAC	GM-family	3566.5219	-3557.5219	-3534.2425
	CL-family	3622.3425	-3613.1	-3590.1
	IG-family	3631.47197	-3622.472	-3599.1926
	GM-GM-GM	3298.2271	-3293.2271	-3280.2941

Note: "TV-" represents time-varying copula models. GM represents Gumbel copula, CL represents Clayton copula, IG represents IG copula, G represents Gamma Lévy exponent, and IG represents Inverse Gaussian Lévy exponent.

Conclusions

- Verify the multi-layer LSHAC models theoretically.
- Introduce a three-stage estimation procedure and provide a new dissimilarity measure by incorporating Euclidean and dependence distances.
- Demonstrate that compared to time-varying elliptical copulas, LS-HAC models provide the best goodness of fit.

Thanks for Attentions!

Six-Dimensional LS-HAC

- $\psi_{0,1}^{(0)}$ (outer generator) : GM or $GM \circ G$

- If $\psi_{0,1}^{(0)}$ is $GM \circ G$ generator and $\tilde{\psi}_{1,1}^{(1)}$ is IG subordinator, we have a five-parameter inner generator

$$\psi_{1,1}^{(1)} = GM \circ G \circ IG$$

$$= \exp\left(- \left(a_1 \log(1 + \frac{a_2}{b_1}(\sqrt{2u + b_2^2} - b_2)) \right)^{\frac{1}{\theta}} \right)$$

- **LS-HAC model is very flexible.**
- ***Note that Gumbel HAC is a special case of LS-HAC when outer generator and subordinator are Gumbel.***

References

[1] Lee, S. and Lin, S. (2010), Modeling and Evaluating Insurance Losses via Mixtres of Erlang Distributions. *North American Actuarial Journal* 14(1), 107-130.

[2] Hans Bühlmann (1980), An Economic Premium Principle. *ASTIN Bullentin* 11, 52-60.

[3] Marius Hofert (2012), A Stochastic Representation and Sampling Algorithm for Nested Archimedean Copulas. *Journal of Computation and Simulation* 82(101), 1239-1255.

[4] Christian Hering and Marius Hofert and Jan-Frederik Mai and Matthias Scherer (2010), Constructing Hierarchical Archimedean Copulas with Lévy Subordinators. *Journal of Multivariate Analysis* 101, 1428-1433.

[5] Jan-Frederik Mai and Matthias Scherer (2012), H-extendible Copulas. *Journal of Multivariate Analysis* 100, 1567-1585.

Generalized Hyperbolic (GH)

- The probability density function of the GH distribution can be shown as follows

$$f_{GH}\left(x\middle|\alpha,\beta,\lambda,\delta,m\right)=\frac{\left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\lambda}}{\sqrt{2\pi}\left(K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)\right)}e^{\beta(z_t-m)}\frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-m)^2}\right)}{\left(\frac{\sqrt{\delta^2+(x-m)^2}}{\alpha}\right)^{\frac{1}{2}-\lambda}},$$

- The moment generating function of GH distribution is given by

$$\phi_{GH}(\omega)=\exp\left(\psi_{GH}(\omega)\right)=\mathrm{e}^{i\omega m}\left(-\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+i\omega)^2}\right)^{\lambda/2}\frac{K_{\lambda}\left(\delta\sqrt{\alpha^2-(\beta+i\omega)^2}\right)}{K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}$$