Levy Subordinated Hierarchical Archimedean Copula: Theory and Application

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(with Ken Seng Tan and Wenjun Zhu)

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• Elliptical dependence structures (Gaussian and Student’s t Copulas) are widely used.

  => do not allow for correlation asymmetries

• Archimedean Copulas (AC) such as Gumbel and Calyton Copula are used to capture the asymmetric tail dependence.
Archimedean copulas

\[ C(u_1, u_2, \ldots, u_d) = \prod \psi[\psi^{-1}(u_1), \ldots, \psi^{-1}(u_d)], \]

is a d-dimensional Archimedean copula iff \( \psi \in \mathcal{G} \) defined as

\[ \left\{ \psi : [0, \infty) \to [0, 1] \mid \psi(0) = 1, \psi(\infty) = 0, (-1)^k \frac{d^k}{du^k} \psi(u) \geq 0, k \in \mathbb{N} \right\} \]

a set of completely monotonic (c.m.) functions.
Drawbacks of AC

Exchangeable

• invariant under permutation

=> inappropriate in the multivariate model.

• Hierarchical Archimedean copula (HAC) has been proposed to overcome the disadvantage of Archimedean Copulas (AC).
3-level, six-dimensional HAC

“Copula of Copulas” Structure

\[ C_{0,1}^{(0)}(u_1, \cdots, u_6) = C_{0,1}^{(0)}(C_{1,1}^{(1)}(C_{1,1}^{(2)}(u_1, u_2), u_3), C_{1,2}^{(1)}(C_{2,1}^{(2)}(u_4, u_5), u_6)) \]
Condition (4) is called **compatible condition**, adding difficulties in constructing HAC.

- Gumbel HAC satisfies Equation (4) and are easy to simulate. As a result, it is the most frequently used HAC in empirical study.
Levy Subordinated HAC

- Hering et al. (2010) first construct a two-level hierarchical model based on Lévy subordinators.

Theorem 2.1:

\[(\psi_0^{-1} \circ \psi_1)' \text{ is c.m. } \iff \psi_0^{-1} \circ \psi_1 = \Psi \text{ is the Laplace exponent of a Lévy subordinator.}\]

- Mai and Scherer (2012), introduced h-extendible copulas including a three-level LS-HAC case.
Main Goals

• Introduce a general framework of LS-HAC models with arbitrary levels.

• Propose a three-stage estimation procedure. We use hierarchical clustering analysis to determine LSHAC structure.

• Empirically examine performance of LS-HAC models.
Figure: General Framework of LS-HAC Model

Outer Copula

Inner Copulas

Multi-layer
Integral Representation

Theorem

Given the general structure of LS-HAC, the copula function can be constructed as the following:

\[
\int_0^\infty \prod_{j_1=1}^\infty \int_0^\infty \prod_{j_2=1}^\infty \cdots \int_0^\infty \prod_{j_{L-1}=1}^\infty \int_0^\infty \prod_{j_L=1}^\infty \left( F_{s_{L-2} j_{L-1}}^{(L-1)} (\bar{u}_{s_{L-1} j_L}) \right)^{v_{L-1}^{(L-1)}} (dG)^{(L-1)}_{j_{L-1}},
\]

where we define \((l = 1, \ldots, L - 1)\):

\[
(dG)^{(0)} = dG^{(0)}_{0,1}(v^{(0)}_{0,1}),
\]

and

\[
(dG)^{(l)}_{j_l} = d\tilde{G}_{s_{l-1} j_l}^{(l)} (v^{(l)}_{s_{l-1} j_l}, v^{(l-1)}_{s_{l-2} j_{l-1}}) \cdots d\tilde{G}_{s_0 j_1}^{(l)} (v^{(1)}_{s_0 j_1}, v^{(0)}_{0,1}) dG^{(0)}_{0,1}(v^{(0)}_{0,1}).
\]
Generators of LS-HAC

Proposition

For Calibration Purpose

For $1 \leq l \leq L$, in level $l$, the $j_l$-th copula generator in position $s_{l-1}$: $\psi_{s_{l-1}j_l}^{(l)}$, can be expressed as:

\[
\psi_{s_{l-1}j_l}^{(l)} = \psi_{0,1}^{(0)} \bigcirc \bigcirc_{i=1}^{l-1} \sim (i+1)_{s_{i}j_{i+1}}
\]

where $\bigcirc_{i=1}^{n} f_i := f_1 \circ \ldots \circ f_n$, and $\psi_{s_{l-1}j_l}^{(l)}$ is c.m..
A three-layer Six-Dimensional LS-HAC

\[
\begin{align*}
\psi_{1,1}^{(1)} &= \psi_{0,1}^{(0)} \circ \tilde{\psi}_{1,1}^{(1)}, \\
\psi_{1,1}^{(2)} &= \psi_{1,1}^{(1)} \circ \tilde{\psi}_{1,1}^{(2)} = \psi_{0,1}^{(0)} \circ \tilde{\psi}_{1,1}^{(1)} \circ \tilde{\psi}_{1,1}^{(2)}
\end{align*}
\]

In the previous expressions, \(\psi_{i,j}^{(l)}\) is AC generators and \(\tilde{\psi}_{s,t}^{(l)}\) is Laplace exponent of Lévy subordinators.
In this paper, we consider

- **AC generators**

- **Clayton (CL)**

- **Gumbel (GM)**

- **Inverse Gaussian (IG)**

<table>
<thead>
<tr>
<th>Name</th>
<th>$\psi(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CL</td>
<td>$(1 + u)^{-\frac{1}{\theta}}, \theta \geq 0$</td>
</tr>
<tr>
<td>GM</td>
<td>$\exp\left(-x^{\frac{1}{\theta}}\right)$</td>
</tr>
<tr>
<td>IG</td>
<td>$\exp\left(\frac{1}{\theta}(1 - \sqrt{1 + 2\theta^2x})\right)$</td>
</tr>
</tbody>
</table>
LS-HAC

• Levy subordinators
  • Gamma (G)
  • Gumbel or Stable (GM)
  • Inverse Gaussian (IG)

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Inverse Gaussian (IG)</th>
<th>Gamma (G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Characteristic Function (c.f.)</td>
<td>$\phi_{IG}(u) = \exp(-\omega(a\sqrt{2ui} + b^2 - ab))$</td>
<td>$\phi_G = \exp(-\omega \log(1 - iu/b))$</td>
</tr>
<tr>
<td>Laplace Exponent</td>
<td>$\Psi_{IG}(u) = a\sqrt{2u + b^2} - ab$</td>
<td>$\Psi_G(u) = a\log(1 + u/b)$</td>
</tr>
</tbody>
</table>

• **Note that Gumbel HAC is a special case of LS-HAC when outer generator and subordinator are Gumbel.**
LS-HAC Estimation

• Three-Stage Estimation
  • Stage 1: Estimating Margins

• Stage 2: Determining Hierarchical Structure
  => Hierarchical Clustering Analysis

• Stage 3: Estimating LS-HAC parameters by using global MLE, instead of using sequential MLE in classical Gumbel HAC.
Stage 1: ARMA-GARCH-GH models

- Autoregressive moving average-generalized autoregressive conditional heteroskedasticity (ARMA-GARCH) models (Bollerslev, 1986; Engle, 1982) are used to analyze the dynamics of the margins for the financial data.

\[
\text{ARMA (m,n)-GARCH (p,q) Model}
\]

\[
r_{i,t} = c_i + \sum_{j=1}^{m} \phi_{i,j} r_{i,t-j} + \sum_{j=1}^{n} \theta_{i,j} \epsilon_{i,t-j} + \epsilon_{i,t},
\]

\[
\epsilon_{i,t} = \sqrt{h_{i,t}} z_{i,t},
\]

\[
h_{i,t} = \omega_i + \sum_{j=1}^{p} a_{i,j} h_{i,t-j} + \sum_{j=1}^{q} b_{i,j} \epsilon_{i,t-j}^2.
\]
Generalized Hyperbolic (GH)

- Barndorff-Nielsen (1977) uses the GH distribution for financial time series exhibiting skewness, leptokurtosis and tail-thickness.

- The GH distribution nests many well-known highly flexible distributions (normal, Student-$t$, hyperbolic, Variance Gamma (VG), Normal Inverse Gaussian and GH skewed $t$ distributions as special case or limit cases.

- After obtaining the marginal parameters, we use probability transform based on the best goodness-of-fit residual distribution to obtain the inputs of copulas.

$$u_{it} = F_{Z_i}(z_{it})$$
Stage 2: Hierarchical Clustering Analysis

- Despite its importance, few papers discuss the grouping method of the HAC or LSHAC models.

- Okhrin et. al. (2013) introduce an estimation procedure to determine optimal structure of HAC by evaluating all possible structures.
  - Time-consuming and inefficient to enumerate all possible structures, especially for high-dimensional problems.

- We employ hierarchical clustering method to determine the structure of LSHAC.
Dissimilarity (Proximity) Matrix

Using six dimension data as an example:

$$
\begin{pmatrix}
  d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \\
  d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & d_{2,6} \\
  d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} & d_{3,6} \\
  d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & d_{4,6} \\
  d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & d_{5,6} \\
  d_{6,1} & d_{6,2} & d_{6,3} & d_{6,4} & d_{6,5} & d_{6,6}
\end{pmatrix}
$$

- $d_{i,j}$ represents dissimilarity of $i$ and $j$.
- $d_{i,i} = 0$ and $d_{i,j} = d_{j,i}$ (symmetric)
- High $d_{i,j}$ represents high degree of dissimilarity

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Euclidean V.S. Dependence Distance:

- Euclidean Distance: 
  \[ d_{ij}^{\text{Euclid}} = \sqrt{\sum_k (x_{k,i} - x_{k,j})^2} \]

- Dependence Distance 
  \[ d_{ij}^{\text{Corr}} = 1 - \mathcal{R}_{i,j} \]

where \( \mathcal{R}_{i,j} \) is an association measure, such as Spearman's rho and Kendall's tau.

- In literature, a sequential procedure according to Kendall's tau is used to determine hierarchical structure of classical Gumbel HAC.

- It ignores dissimilarity from “Euclidean Distance” .

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Counter Example

• This example shows the problem of the two distances.

\[
\begin{align*}
x_1 &= \text{rand}(1, 100), \\
x_2 &= 0.051 : 0.001 : 0.15, \\
x_3 &= 0.001 : 0.001 : 0.10, \\
x_4 &= 0.851 : 0.001 : 0.95, \\
x_5 &= 0.9 \times (0.801 : 0.001 : 0.90)
\end{align*}
\]

• Intuitively, we have
  • \(x_2\) and \(x_3\) => perfect positive correlated with small values
  • \(x_4\) and \(x_5\) => perfect positive correlated with large values
  • \(x_1\) => independent of others (third group)
Euclidean Distance

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Kendall's Tau measure

It groups $x_2$ to $x_5$ as a *perfect-positive-correlated* group without further discrimination.
A Trade-off measure

- In this paper, we propose a new measure by dividing Euclidean distance by dependence distances

\[
d_{i,j}^{\text{Euclid}} \div \frac{1 + \tau_{i,j}}{1}
\]
Empirical Analysis

- Data: 5-year daily data of Stock indices
  - European (France, UK and Germany)
  - American (Brazil, Canada and US)
Stage 1: Margins

Marginal Estimation Results (GARCH(1,1) effect but no ARMA effect)

<table>
<thead>
<tr>
<th>Indices</th>
<th>IBOV</th>
<th>NDX</th>
<th>SPTSX</th>
<th>CAC</th>
<th>UKX</th>
<th>DAX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best Distribution</td>
<td>t</td>
<td>VG</td>
<td>VG</td>
<td>t</td>
<td>VG</td>
<td>VG</td>
</tr>
<tr>
<td>$c \times 10^{-4}$</td>
<td>1.7676 (0.3753)</td>
<td>10.4844 (3.8384)**</td>
<td>2.4634 (0.8901)</td>
<td>9.5697 (2.4141)**</td>
<td>4.5534 (1.5001)</td>
<td>8.8959 (2.2612)**</td>
</tr>
<tr>
<td>$\omega \times 10^{-6}$</td>
<td>5.617 (2.227)**</td>
<td>2.9194 (2.7788)**</td>
<td>3.9638 (0.9451)</td>
<td>2.7935 (1.526)*</td>
<td>8.2843 (1.2079)</td>
<td>1.3621 (1.1318)</td>
</tr>
<tr>
<td>$a_1$</td>
<td>0.9191 (53.6884)**</td>
<td>0.8775 (36.0184)**</td>
<td>0.9302 (62.5136)**</td>
<td>0.9254 (46.6514)**</td>
<td>0.9374 (51.7158)**</td>
<td>0.9363 (58.6861)**</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.0685 (4.3703)**</td>
<td>0.1048 (4.5191)**</td>
<td>0.0698 (4.3598)**</td>
<td>0.0683 (3.5849)**</td>
<td>0.0586 (3.3877)**</td>
<td>0.0597 (3.9366)**</td>
</tr>
<tr>
<td>$\nu$</td>
<td>8 (6.7306)**</td>
<td></td>
<td></td>
<td></td>
<td>6.5103 (3.5849)**</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.9846 (11.8214)**</td>
<td>2.4289 (7.3098)**</td>
<td></td>
<td>2.4826 (7.5641)**</td>
<td>2.174 (9.3175)**</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>-0.1753 (11.8214)**</td>
<td>-0.415 (-3.3496)**</td>
<td></td>
<td>-0.1739 (-1.9242)**</td>
<td>-0.1739 (-1.9242)**</td>
<td></td>
</tr>
</tbody>
</table>
Empirical Analysis: Hierarchical clustering

Dendrogram according to our distance measure

Geographic Groups

European stock indices

American stock indices

CAC       DAX       UKX       IBOV       NASDAQ100      SPTSX

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Empirical Analysis: Global MLE

Table: Six Dimension Estimating Results

<table>
<thead>
<tr>
<th>Copula</th>
<th>Log-Likelihood</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>3361.4681</td>
<td>-3346.4681</td>
<td>-3309.3370</td>
</tr>
<tr>
<td>TV-Gaussian</td>
<td>3384.0908</td>
<td>-3367.0908</td>
<td>-3325.0089</td>
</tr>
<tr>
<td>T</td>
<td>3558.7891</td>
<td>-3542.7891</td>
<td>-3503.1826</td>
</tr>
<tr>
<td>TV-T</td>
<td>3573.9428</td>
<td>-3555.9428</td>
<td>-3511.3855</td>
</tr>
<tr>
<td>GM-family</td>
<td>3566.5219</td>
<td>-3557.5219</td>
<td>-3534.2425</td>
</tr>
<tr>
<td>CL-family</td>
<td>3622.3425</td>
<td>-3613.1</td>
<td>-3590.1</td>
</tr>
<tr>
<td>IG-family</td>
<td>3631.47197</td>
<td>-3622.472</td>
<td>-3599.1926</td>
</tr>
<tr>
<td>GM-GM-GM</td>
<td>3298.2271</td>
<td>-3293.2271</td>
<td>-3280.2941</td>
</tr>
</tbody>
</table>

Note: “TV-” represents time-varying copula models. GM represents Gumbel copula, CL represents Clayton copula, IG represents IG copula, G represents Gamma Lévy exponent, and IG represents Inverse Gaussian Lévy exponent.
Conclusions

- Verify the multi-layer LSHAC models theoretically.

- Introduce a three-stage estimation procedure and provide a new dissimilarity measure by incorporating Euclidean and dependence distances.

- Demonstrate that compared to time-varying elliptical copulas, LS-HAC models provide the best goodness of fit.
Thanks for Attentions!
Six-Dimensional LS-HAC

- \( \psi_{0,1}^{(0)} \) (outer generator) : \( GM \) or \( GM \circ G \)

- If \( \psi_{0,1}^{(0)} \) is \( GM \circ G \) generator and \( \tilde{\psi}_{1,1}^{(1)} \) is IG subordinator, we have a five-parameter inner generator

\[
\psi_{1,1}^{(1)} = GM \circ G \circ IG
\]

\[
= \exp \left( - \left( a_1 \log(1 + \frac{a_2}{b_1} \sqrt{2u + b_2^2 - b_2}) \right)^{\frac{1}{\psi}} \right)
\]

- LS-HAC model is very flexible.

- Note that Gumbel HAC is a special case of LS-HAC when outer generator and subordinator are Gumbel.
References


Generalized Hyperbolic (GH)

- The probability density function of the GH distribution can be shown as follows

\[
f_{GH}(x | \alpha, \beta, \lambda, \delta, m) = \frac{\left(\frac{\sqrt{\alpha^2 - \beta^2}}{\delta}\right)^\lambda}{\sqrt{2\pi} \left(K_\lambda \left(\delta \sqrt{\alpha^2 - \beta^2}\right)\right)} e^{\beta(x - m)} K_{\lambda - \frac{1}{2}} \left(\frac{\alpha \sqrt{\delta^2 + (x - m)^2}}{\delta^{1/2}}\right),
\]

- The moment generating function of GH distribution is given by

\[
\phi_{GH}(\omega) = \exp \left(\psi_{GH}(\omega)\right) = e^{i\omega m} \left(-\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + i\omega)^2}\right)^{\lambda/2} \frac{K_\lambda \left(\delta \sqrt{\alpha^2 - (\beta + i\omega)^2}\right)}{K_{\lambda - \frac{1}{2}} \left(\delta \sqrt{\alpha^2 - \beta^2}\right)}
\]