Optimal Dividends in the Dual Model under Transaction Costs

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De Finetti's Dividend Problem

Given a stochastic process X, the problem is to choose a nondecreasing process L so as to maximize

$$\mathbb{E}\left[\int_0^{\sigma} e^{-qt} \mathrm{d}L_t\right]$$

where $\sigma := \inf\{t > 0 : X_t - L_t < 0\}$.

Some History

- Random walk model by De Finetti (1957).
- Brownian motion model by Jeanblanc and Shiryaev (1995).
- Spectrally negative Lévy models
  - Use of fluctuation/excursion theories and scale functions pioneered by, e.g. Bertoin, Doney & Kyprianou.
- Spectrally positive Lévy models
  - Avanzi, Gerber & Shiu (IME, 2007) and Avanzi & Gerber (ASTIN, 2008) (focus on i.i.d. (hyper)exponential jumps).
  - Bayraktar, Kyprianou & Y. (ASTIN, 2013) for a general spectrally positive Lévy process.
  - This paper extends the above by introducing fixed transaction costs.

Outline

- Solutions to the dual model
  - with review on fluctuation theory of (reflected) Lévy processes.
- Extensions with transaction costs (impulse control):

$$v_\pi(x) := \mathbb{E}_x\Big[\int_0^{\sigma^\pi} e^{-qt} \mathrm{d}\Big(L_t^\pi - \sum_{0 \leq s < t} \beta 1_{\{\Delta L_s^\pi > 0\}}\Big)\Big].$$

- Numerical Results.

Spectrally Positive Lévy Processes

- Let X be a spectrally positive Lévy process with Laplace exponent:

$$\begin{aligned}\psi(s) &:= \log \mathbb{E}\left[e^{-sX_1}\right] \\ &= cs + \frac{1}{2}\sigma^2 s^2 + \int_{(0,\infty)} (e^{-sz} - 1 + sz1_{\{0<z<1\}})\nu(\mathrm{d}z),\end{aligned}$$

  such that $\int_{(0,\infty)}(1 \wedge z^2)\nu(\mathrm{d}z) < \infty$.
- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(0,1)} z\,\nu(\mathrm{d}z) < \infty$.
- We exclude the trivial case in which X is a subordinator.

The Classical Dual Model

- A strategy $\pi = \{L_t^\pi, t \geq 0\}$ is a nondecreasing, right-continuous and adapted process starting at zero.
- A controlled risk process is the difference:

$$U_t^\pi := X_t - L_t^\pi, \quad t \geq 0.$$

- Time of ruin: $\sigma^\pi := \inf\{t > 0 : U_t^\pi < 0\}$.
- We want to maximize, for $q > 0$,

$$v_\pi(x) := \mathbb{E}_x\left[\int_0^{\sigma^\pi} e^{-qt} \mathrm{d}L_t^\pi\right],$$

over the set of all strategies Π satisfying $L_t^\pi - L_{t-}^\pi \leq U_{t-}^\pi + \Delta X_t$ for all $t \leq \sigma^\pi$ *a.s.*
- We want to obtain the value function:

$$v(x) := \sup_{\pi \in \Pi} v_\pi(x), \quad x \geq 0.$$

Solution Procedures

- We follow a classical approach "guess" and "verify".
- Guess that an optimal strategy is a barrier strategy (reflected Lévy process) $\pi_a := \{L_t^a; t \leq \sigma_a\}$ in the form:

$$L_t^a := \sup_{0 \leq s \leq t} (X_s - a) \vee 0,$$
$$U_t^a := X_t - L_t^a,$$

with the corresponding ruin time $\sigma_a := \inf\{t > 0 : U_t^a < 0\}$.

- Choose the value of a using some smoothness condition.
- Verify that

$$v_a(x) := \mathbb{E}_x \left[\int_0^{\sigma_a} e^{-qt} \mathrm{d} L_t^a \right] \geq \sup_{\pi \in \Pi} v_\pi(x).$$

Scale Functions

- Recall that X is a spectrally positive Lévy process with Laplace exponent $\psi(s) = \log \mathbb{E}\left[e^{-sX_1}\right]$.
- Fix any $q > 0$, there exists a function called the q-scale function

$$W^{(q)} : \mathbb{R} \to [0, \infty),$$

which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0, \infty)$, and is characterized by the Laplace transform:

$$\int_0^{\infty} e^{-sx} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(s) - q}, \qquad s > \Phi(q),$$

where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

Scale Functions

Let us define the first down- and up-crossing times, respectively, by

$$\tau_a^- := \inf\{t \geq 0 : X_t < a\}$$
$$\tau_b^+ := \inf\{t \geq 0 : X_t > b\}.$$

Then we have for any $b > 0$

$$\mathbb{E}_x\left[e^{-q\tau_0^-} 1_{\{\tau_b^+ > \tau_0^-\}}\right] = \frac{W^{(q)}(b-x)}{W^{(q)}(b)},$$
$$\mathbb{E}_x\left[e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}}\right] = Z^{(q)}(b-x) - Z^{(q)}(b)\frac{W^{(q)}(b-x)}{W^{(q)}(b)},$$

where

$$Z^{(q)}(x) := 1 + q\overline{W}^{(q)}(x),$$
$$\overline{W}^{(q)}(x) := \int_0^x W^{(q)}(y)\mathrm{d}y.$$

Main Results

Let $\mu := \mathbb{E}X_1$. We will denote our candidate barrier level by

$$a^* = \begin{cases} \left(\overline{Z}^{(q)}\right)^{-1}\left(\frac{\mu}{q}\right) > 0 & \text{if } \mu > 0, \\ 0 & \text{if } \mu \leq 0, \end{cases}$$

which is well-defined because $\overline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(z)\mathrm{d}z$ is monotone.

Theorem (Bayraktar, Kyprianou & Y. (Astin Bull., 2013))

We have

$$v_{a^*}(x) := \sup_{\pi \in \Pi} v_\pi(x), \quad x \geq 0,$$

where

$$v_{a^*}(x) = \begin{cases} -\overline{Z}^{(q)}(a^* - x) - \frac{\psi'(0+)}{q} = -\overline{Z}^{(q)}(a^* - x) + \frac{\mu}{q}, & \text{if } \mu > 0, \\ x, & \text{if } \mu \leq 0. \end{cases}$$

Extension with Transaction Costs

- Consider an extension where we want to maximize

$$v_\pi(x) := \mathbb{E}_x \Big[\int_0^{\sigma^\pi} e^{-qt} \mathrm{d}\Big(L_t^\pi - \sum_{0 \le s < t} \beta 1_{\{\Delta L_s^\pi > 0\}} \Big) \Big],$$

  for some fixed unit transaction cost $\beta > 0$.
- A strategy $\pi = \{L_t^\pi, t \geq 0\}$ is assumed to be a pure-jump, nondecreasing, right-continuous and adapted process starting at zero.

The (c_1, c_2)-policy

- For $c_2 > c_1 \geq 0$, a (c_1, c_2)-policy, $\pi_{c_1,c_2} := \left\{ L_t^{c_1,c_2}; t \geq 0 \right\}$, brings the level of the controlled risk process $U^{c_1,c_2} := X - L^{c_1,c_2}$ down to c_1 whenever it reaches or exceeds c_2.
- We aim to prove that a (c_1^*, c_2^*)-policy is optimal for some $c_2^* > c_1^* \geq 0$.
- We shall express, in terms of the scale function,

$$v_{c_1,c_2}(x) := \mathbb{E}_x \Big[\int_0^{\sigma_{c_1,c_2}} e^{-qt} \mathrm{d} \Big(L_t^{c_1,c_2} - \sum_{0 \leq s < t} \beta 1_{\{\Delta L_s^{c_1,c_2} > 0\}} \Big) \Big],$$

  where $\sigma_{c_1,c_2} := \inf \left\{ t > 0 : U_t^{c_1,c_2} < 0 \right\}$ is the corresponding ruin time.

Computing v_{c_1,c_2}

- By the strong Markov property, it must satisfy for every $0 \leq x \leq c_2$ and $0 \leq c_1 < c_2$

$$v_{c_1,c_2}(x) = \mathbb{E}_x \left[e^{-q\tau_{c_2}^+} 1_{\{\tau_{c_2}^+ < \tau_0^-\}} (X_{\tau_{c_2}^+} - c_1 - \beta) \right] + \mathbb{E}_x \left[e^{-q\tau_{c_2}^+} 1_{\{\tau_{c_2}^+ < \tau_0^-\}} \right] \bar{v}_{c_1,c_2},$$

where $\bar{v}_{c_1,c_2} := v_{c_1,c_2}(c_1)$.

- Solving for $x = c_1$, we have

$$\bar{v}_{c_1,c_2} = \frac{\mathbb{E}_{c_1} \left[e^{-q\tau_{c_2}^+} 1_{\{\tau_{c_2}^+ < \tau_0^-\}} (X_{\tau_{c_2}^+} - c_1 - \beta) \right]}{1 - \mathbb{E}_{c_1} \left[e^{-q\tau_{c_2}^+} 1_{\{\tau_{c_2}^+ < \tau_0^-\}} \right]}, \quad 0 \leq c_1 < c_2.$$

- These can be rewritten in terms of the scale function.

Candidate Value function

Lemma (Bayraktar, Kyprianou & Y. (IME, forthcoming))

For $0 < x < c_2$ *and* $0 \leq c_1 < c_2$,

$$v_{c_1,c_2}(x) = -\overline{Z}^{(q)}(c_2 - x) + \frac{\mu}{q} + \gamma(c_1, c_2)Z^{(q)}(c_2 - x) - G(c_1, c_2)\frac{W^{(q)}(c_2 - x)}{W^{(q)}(c_2)},$$

where

$$\gamma(c_1, c_2) := \bar{v}_{c_1,c_2} + c_2 - c_1 - \beta - \frac{\mu}{q},$$
$$G(c_1, c_2) := \gamma(c_1, c_2)Z^{(q)}(c_2) - \overline{Z}^{(q)}(c_2) + \frac{\mu}{q}.$$

For $x \geq c_2$, $v_{c_1,c_2}(x) = x - c_1 - \beta + \bar{v}_{c_1,c_2}$.

Solution Procedures

- Choose $0 \leq c_1^* < c_2^*$ such that $v_{c_1,c_2}(x)$ (or $\bar{v}_{c_1,c_2} - c_1$) is maximized.
- Examine the smoothness of $v_{c_1^*,c_2^*}(x)$; it turns out that
  - at $x = c_2^*$, $v_{c_1^*,c_2^*}(x)$ is C^0 (C^1) when X is of bounded (unbounded) variation;
  - at $x = c_1^*$, $v'_{c_1^*,c_2^*}(c_1^*) = 1$ when $c_1^* > 0$ and $v'_{c_1^*,c_2^*}(c_1^*) \leq 1$ when $c_1^* = 0$.
- Verify the optimality.
- Show uniqueness of (c_1^*, c_2^*).

Main Results

Theorem (Bayraktar, Kyprianou & Y. (IME, 2013))

- *There exist* unique $0 \le c_1^* < c_2^* < \infty$ *such that* $G(c_1^*, c_2^*) = 0$ *and either*

  case 1 $c_1^* > 0$ *with* $H(c_1^*, c_2^*) = 0$, *or*
  case 2 $c_1^* = 0$ *with* $H(c_1^*, c_2^*) \ge 0$,

  where

$$G(c_1, c_2) := \gamma(c_1, c_2) Z^{(q)}(c_2) - \overline{Z}^{(q)}(c_2) + \frac{\mu}{q},$$
$$H(c_1, c_2) := q \left[\gamma(c_1, c_2) W^{(q)}(c_2 - c_1) - \overline{W}^{(q)}(c_2 - c_1) \right].$$

- *The* (c_1^*, c_2^*)*-strategy is optimal and the value function is*

$$v_{c_1^*, c_2^*}(x) = -\overline{Z}^{(q)}(c_2^* - x) + \frac{\mu}{q} + \gamma(c_1^*, c_2^*) Z^{(q)}(c_2^* - x), \quad x \ge 0.$$

Some properties

Proposition

- *Let* v^{β} *denote the value function corresponding to the dividend payment problem when the fixed transaction cost is* β *(defined as above), and*
- $\hat{v}$ *the value function when there are no-transaction costs.*

Then v^{β} *converges to* $\hat{v}$ *uniformly as* $\beta \downarrow 0$.

Proposition

If $\mu := \mathbb{E}X_1 \leq 0$, *we must have* $c_1^* = 0$.

Numerical Results

Suppose

$$X_t - X_0 = -\mathfrak{d}t + \sigma B_t + \sum_{n=1}^{N_t} Z_n, \quad 0 \le t < \infty,$$

for some $\mathfrak{d} \in \mathbb{R}$ and $\sigma \ge 0$. Here

- $B = \{B_t; t \ge 0\}$ is a standard Brownian motion,
- $N = \{N_t; t \ge 0\}$ is a Poisson process with arrival rate λ, and
- $Z = \{Z_n; n = 1, 2, \ldots\}$ is an i.i.d. sequence of phase-type-distributed r.v. with representation $(m, \boldsymbol{\alpha}, \mathbf{T})$.

Its Laplace exponent is then

$$\psi(s) = \mathfrak{d}s + \frac{1}{2}\sigma^2 s^2 + \lambda\left(\boldsymbol{\alpha}(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t} - 1\right).$$

Numerical Results

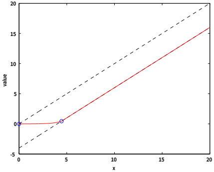
- Suppose $\{-\xi_{i,q}; i \in \mathcal{I}_q\}$ is the set of the roots of the equality $\psi(s) = q$ with negative real parts.
- If these are assumed distinct, the scale function can be written

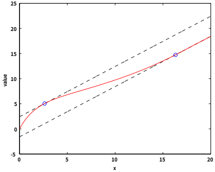
$$W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[e^{\Phi(q)x} - e^{-\xi_{i,q}x} \right], \quad \sigma > 0,$$

$$W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[e^{\Phi(q)x} - e^{-\xi_{i,q}x} \right] + \frac{1}{\mathfrak{d}} e^{\Phi(q)x}, \quad \sigma = 0.$$

- Here $\{\xi_{i,q}; i \in \mathcal{I}_q\}$ and $\{C_{i,q}; i \in \mathcal{I}_q\}$ are possibly complex-valued.
- We choose $(m, \alpha, \mathbf{T})$ to approximate the Weibull distribution with density function with $\alpha = 2$ and $\gamma = 1$ (obtained using the EM-algorithm).

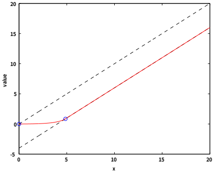
Value functions

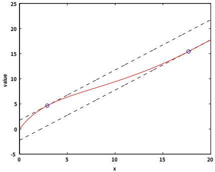




$c_1^* = 0$ $\qquad$ $c_1^* > 0$

without BM

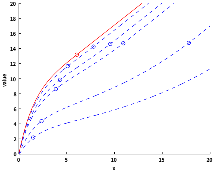




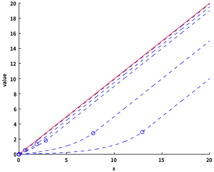
$c_1^* = 0$ $\qquad$ $c_1^* > 0$

with BM

Convergence as $\beta \downarrow 0$

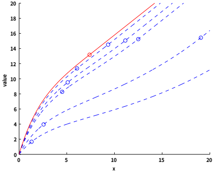


$\mu > 0$

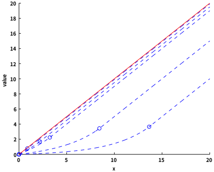


$\mu < 0$

without BM



$\mu > 0$



$\mu < 0$

with BM

References

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[2] E. Bayraktar, A. E. Kyprianou and K. Yamazaki *Optimal Dividends in the Dual Model under Transaction Costs.* Insurance: Mathematics and Economics, 54:133-143, 2014.

[3] M. Egami and K. Yamazaki *Phase-type Fitting of Scale Functions for Spectrally Negative Levy Processes* ,Journal of Computational and Applied Mathematics, 264:1-22, 2014.