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De Finetti’s Dividend Problem

Given a stochastic process $X$, the problem is to choose a nondecreasing process $L$ so as to maximize

$$\mathbb{E} \left[ \int_0^\sigma e^{-qt} dL_t \right]$$

where $\sigma := \inf \{ t > 0 : X_t - L_t < 0 \}$. 
Some History

- Random walk model by De Finetti (1957).
- Spectrally negative Lévy models
  - Use of fluctuation/excursion theories and scale functions pioneered by, e.g. Bertoin, Doney & Kyprianou.
- Spectrally positive Lévy models
  - Avanzi, Gerber & Shiu (IME, 2007) and Avanzi & Gerber (ASTIN, 2008) (focus on i.i.d. (hyper)exponential jumps).
  - Bayraktar, Kyprianou & Y. (ASTIN, 2013) for a general spectrally positive Lévy process.
  - This paper extends the above by introducing fixed transaction costs.
Outline

- Solutions to the dual model
  - with review on fluctuation theory of (reflected) Lévy processes.
- Extensions with transaction costs (impulse control):

  \[ v_\pi(x) := \mathbb{E}_x \left[ \int_0^{\sigma_\pi} e^{-q t} d\left( L^\pi_t - \sum_{0 \leq s < t} \beta 1_{\{\Delta L^\pi_s > 0\}} \right) \right] . \]

- Numerical Results.
Spectrally Positive Lévy Processes

Let $X$ be a spectrally positive Lévy process with Laplace exponent:

$$
\psi(s) := \log \mathbb{E} \left[ e^{-sX_1} \right]
= cs + \frac{1}{2} \sigma^2 s^2 + \int_{(0,\infty)} \left( e^{-sz} - 1 + sz1_{\{0<z<1\}} \right) \nu(dz),
$$

such that $\int_{(0,\infty)} (1 \land z^2) \nu(dz) < \infty$.

It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(0,1)} z \nu(dz) < \infty$.

We exclude the trivial case in which $X$ is a subordinator.
The Classical Dual Model

- A strategy $\pi = \{L^\pi_t, t \geq 0\}$ is a nondecreasing, right-continuous and adapted process starting at zero.
- A controlled risk process is the difference:
  \[ U^\pi_t := X_t - L^\pi_t, \quad t \geq 0. \]
- Time of ruin: $\sigma^\pi := \inf \{ t > 0 : U^\pi_t < 0 \}$.
- We want to maximize, for $q > 0$,
  \[ v_\pi(x) := \mathbb{E}_x \left[ \int_0^{\sigma^\pi} e^{-qt} dL^\pi_t \right], \]
  over the set of all strategies $\Pi$ satisfying $L^\pi_t - L^\pi_{t-} \leq U^\pi_t + \Delta X_t$ for all $t \leq \sigma^\pi$ a.s.
- We want to obtain the value function:
  \[ v(x) := \sup_{\pi \in \Pi} v_\pi(x), \quad x \geq 0. \]
Solution Procedures

- We follow a classical approach “guess” and “verify”.
- Guess that an optimal strategy is a barrier strategy (reflected Lévy process) \( \pi_a := \{L^a_t; t \leq \sigma_a\} \) in the form:

\[
L^a_t := \sup_{0 \leq s \leq t} (X_s - a) \vee 0, \\
U^a_t := X_t - L^a_t,
\]

with the corresponding ruin time \( \sigma_a := \inf \{t > 0 : U^a_t < 0\} \).
- Choose the value of \( a \) using some smoothness condition.
- Verify that

\[
\nu_a(x) := \mathbb{E}_x \left[ \int_0^{\sigma_a} e^{-qt} \, dL^a_t \right] \geq \sup_{\pi \in \Pi} \nu_\pi(x).
\]
Scale Functions

- Recall that $X$ is a spectrally positive Lévy process with Laplace exponent $\psi(s) = \log \mathbb{E} [e^{-sX_1}]$.
- Fix any $q > 0$, there exists a function called the $q$-scale function $W(q) : \mathbb{R} \to [0, \infty)$, which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0, \infty)$, and is characterized by the Laplace transform:

$$\int_0^\infty e^{-sx} W(q)(x)dx = \frac{1}{\psi(s) - q}, \quad s > \Phi(q),$$

where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$. 

Scale Functions

Let us define the first down- and up-crossing times, respectively, by

\[
\tau_a^- := \inf \{ t \geq 0 : X_t < a \}
\]
\[
\tau_b^+ := \inf \{ t \geq 0 : X_t > b \}.
\]

Then we have for any \( b > 0 \)

\[
\mathbb{E}_x \left[ e^{-q\tau_0^-} 1_{\{\tau_b^+ > \tau_0^-\}} \right] = \frac{W(q)(b - x)}{W(q)(b)},
\]
\[
\mathbb{E}_x \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}} \right] = Z(q)(b - x) - Z(q)(b) \frac{W(q)(b - x)}{W(q)(b)},
\]

where

\[
Z(q)(x) := 1 + q\bar{W}(q)(x),
\]
\[
\bar{W}(q)(x) := \int_0^x W(q)(y)dy.
\]
Main Results

Let $\mu := \mathbb{E}X_1$. We will denote our candidate barrier level by

$$a^* = \begin{cases} (\overline{Z}(q))^{-1} \left( \frac{\mu}{q} \right) > 0 & \text{if } \mu > 0, \\ 0 & \text{if } \mu \leq 0, \end{cases}$$

which is well-defined because $\overline{Z}(q)(x) := \int_0^x Z(q)(z)dz$ is monotone.

Theorem (Bayraktar, Kyprianou & Y. (Astin Bull., 2013))

We have

$$v_{a^*}(x) := \sup_{\pi \in \Pi} v_{\pi}(x), \quad x \geq 0,$$

where

$$v_{a^*}(x) = \begin{cases} -\overline{Z}(q)(a^* - x) - \frac{\psi'(0+)}{q} = -\overline{Z}(q)(a^* - x) + \frac{\mu}{q}, & \text{if } \mu > 0, \\ x, & \text{if } \mu \leq 0. \end{cases}$$
Consider an extension where we want to maximize

$$v_\pi(x) := \mathbb{E}_x \left[ \int_0^{\sigma^\pi} e^{-qt} \left( L_t^\pi - \sum_{0 \leq s < t} \beta 1_{\{\Delta L_s^\pi > 0\}} \right) \right],$$

for some fixed unit transaction cost $\beta > 0$.

A strategy $\pi = \{L_t^\pi, t \geq 0\}$ is assumed to be a pure-jump, nondecreasing, right-continuous and adapted process starting at zero.
The \((c_1, c_2)\)-policy

- For \(c_2 > c_1 \geq 0\), a \((c_1, c_2)\)-policy, \(\pi_{c_1, c_2} := \{L_t^{c_1, c_2} ; t \geq 0\}\), brings the level of the controlled risk process \(U_{c_1, c_2} := X - L_{c_1, c_2}\) down to \(c_1\) whenever it reaches or exceeds \(c_2\).

- We aim to prove that a \((c_1^*, c_2^*)\)-policy is optimal for some \(c_2^* > c_1^* \geq 0\).

- We shall express, in terms of the scale function,

\[
\nu_{c_1, c_2}(x) := \mathbb{E}_x \left[ \int_{0}^{\sigma_{c_1, c_2}} e^{-qt} d \left( L_t^{c_1, c_2} - \sum_{0 \leq s < t} \beta 1_{\{\Delta L_s^{c_1, c_2} > 0\}} \right) \right],
\]

where \(\sigma_{c_1, c_2} := \inf \{ t > 0 : U_t^{c_1, c_2} < 0 \}\) is the corresponding ruin time.
Computing $v_{c_1, c_2}$

- By the strong Markov property, it must satisfy for every $0 \leq x \leq c_2$ and $0 \leq c_1 < c_2$

$$v_{c_1, c_2}(x) = \mathbb{E}_x \left[ e^{-q \tau_{c_2}^+} 1_{\{\tau_{c_2}^+ < \tau_{0}^-\}} (X_{\tau_{c_2}^+} - c_1 - \beta) \right]$$

$$+ \mathbb{E}_x \left[ e^{-q \tau_{c_2}^+} 1_{\{\tau_{c_2}^+ < \tau_{0}^-\}} \right] \bar{v}_{c_1, c_2},$$

where $\bar{v}_{c_1, c_2} := v_{c_1, c_2}(c_1)$.

- Solving for $x = c_1$, we have

$$\bar{v}_{c_1, c_2} = \frac{\mathbb{E}_{c_1} \left[ e^{-q \tau_{c_2}^+} 1_{\{\tau_{c_2}^+ < \tau_{0}^-\}} (X_{\tau_{c_2}^+} - c_1 - \beta) \right]}{1 - \mathbb{E}_{c_1} \left[ e^{-q \tau_{c_2}^+} 1_{\{\tau_{c_2}^+ < \tau_{0}^-\}} \right]}, \quad 0 \leq c_1 < c_2.$$  

- These can be rewritten in terms of the scale function.
Candidate Value function

Lemma (Bayraktar, Kyprianou & Y. (IME, forthcoming))

For $0 < x < c_2$ and $0 \leq c_1 < c_2$,

$$v_{c_1,c_2}(x) = -\overline{Z}^{(q)}(c_2 - x) + \frac{\mu}{q} + \gamma(c_1, c_2)Z^{(q)}(c_2 - x)$$

$$- G(c_1, c_2) \frac{W^{(q)}(c_2 - x)}{W^{(q)}(c_2)},$$

where

$$\gamma(c_1, c_2) := \bar{v}_{c_1,c_2} + c_2 - c_1 - \beta - \frac{\mu}{q},$$

$$G(c_1, c_2) := \gamma(c_1, c_2)Z^{(q)}(c_2) - \overline{Z}^{(q)}(c_2) + \frac{\mu}{q}.$$

For $x \geq c_2$, $v_{c_1,c_2}(x) = x - c_1 - \beta + \bar{v}_{c_1,c_2}$. 
Solution Procedures

■ Choose $0 \leq c_1^* < c_2^*$ such that $v_{c_1, c_2}(x)$ (or $\bar{v}_{c_1, c_2} - c_1$) is maximized.

■ Examine the smoothness of $v_{c_1^*, c_2^*}(x)$; it turns out that
  - at $x = c_2^*$, $v_{c_1^*, c_2^*}(x)$ is $C^0$ ($C^1$) when $X$ is of bounded (unbounded) variation;
  - at $x = c_1^*$, $v'_{c_1^*, c_2^*}(c_1^*) = 1$ when $c_1^* > 0$ and $v'_{c_1^*, c_2^*}(c_1^*) \leq 1$ when $c_1^* = 0$.

■ Verify the optimality.

■ Show uniqueness of $(c_1^*, c_2^*)$. 

Main Results

Theorem (Bayraktar, Kyprianou & Y. (IME, 2013))

There exist unique \(0 \leq c_1^* < c_2^* < \infty\) such that \(G(c_1^*, c_2^*) = 0\) and either

- **case 1** \(c_1^* > 0\) with \(H(c_1^*, c_2^*) = 0\), or
- **case 2** \(c_1^* = 0\) with \(H(c_1^*, c_2^*) \geq 0\),

where

\[
G(c_1, c_2) := \gamma(c_1, c_2)Z(q)(c_2) - Z(q)(c_2) + \frac{\mu}{q},
\]

\[
H(c_1, c_2) := q \left[ \gamma(c_1, c_2)W(q)(c_2 - c_1) - W(q)(c_2 - c_1) \right].
\]

The \((c_1^*, c_2^*)\)-strategy is optimal and the value function is

\[
v_{c_1^*, c_2^*}(x) = -Z(q)(c_2^* - x) + \frac{\mu}{q} + \gamma(c_1^*, c_2^*)Z(q)(c_2^* - x), \quad x \geq 0.
\]
Some properties

Proposition

- Let $v^\beta$ denote the value function corresponding to the dividend payment problem when the fixed transaction cost is $\beta$ (defined as above), and
- $\hat{v}$ the value function when there are no-transaction costs.

Then $v^\beta$ converges to $\hat{v}$ uniformly as $\beta \downarrow 0$.

Proposition

If $\mu := \mathbb{E}X_1 \leq 0$, we must have $c_1^* = 0$. 
Numerical Results

Suppose

\[ X_t - X_0 = -\vartheta t + \sigma B_t + \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty, \]

for some \( \vartheta \in \mathbb{R} \) and \( \sigma \geq 0 \). Here

- \( B = \{B_t; t \geq 0\} \) is a standard Brownian motion,
- \( N = \{N_t; t \geq 0\} \) is a Poisson process with arrival rate \( \lambda \), and
- \( Z = \{Z_n; n = 1, 2, \ldots\} \) is an i.i.d. sequence of phase-type-distributed r.v. with representation \((m, \alpha, T)\).

Its Laplace exponent is then

\[ \psi(s) = \vartheta s + \frac{1}{2}\sigma^2 s^2 + \lambda (\alpha(sI - T)^{-1}t - 1). \]
Numerical Results

- Suppose \( \{-\xi_{i,q}; i \in \mathcal{I}_q\} \) is the set of the roots of the equality \( \psi(s) = q \) with negative real parts.

- If these are assumed distinct, the scale function can be written

\[
W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[ e^{\Phi(q)x} - e^{-\xi_{i,q}x} \right], \quad \sigma > 0,
\]

\[
W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[ e^{\Phi(q)x} - e^{-\xi_{i,q}x} \right] + \frac{1}{\sigma} e^{\Phi(q)x}, \quad \sigma = 0.
\]

- Here \( \{\xi_{i,q}; i \in \mathcal{I}_q\} \) and \( \{C_{i,q}; i \in \mathcal{I}_q\} \) are possibly complex-valued.

- We choose \((m, \alpha, T)\) to approximate the Weibull distribution with density function with \( \alpha = 2 \) and \( \gamma = 1 \) (obtained using the EM-algorithm).
Value functions

\[ c_1^* = 0 \quad \text{without BM} \]

\[ c_1^* > 0 \quad \text{with BM} \]
Convergence as $\beta \downarrow 0$

$\mu > 0$

without BM

$\mu > 0$

with BM

$\mu < 0$
References

